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THEORY OF NUMBERS

Dickson, Leonard Eugene. **Modern Elementary Theory of Numbers.** University of Chicago Press, Chicago, 1939. vii+309 pp. $3.00.

The first four chapters of this book furnish a brief but satisfactory introduction to the usual elementary topics of number theory, including a short account of binary quadratic forms. This part of the book can be easily read by a beginner and there are many problems suitable for such a reader. The remaining two hundred pages deal with more advanced subjects, most of them "additive" in character. The proofs are "elementary" (except for the Appendix) and are, in general, simpler than those in the literature. There are several chapters dealing with quadratic forms. One of these contains generalizations of the classical theorem on representing a natural number as the sum of three squares. Application is made to certain "universal" forms, such as $x^2+y^2+z^2+s\cdot w^2$ for $s=1, 2, \cdots, 7$, each of which is capable of representing every positive integer. Closely connected with this is a chapter which is built around Fermat's conjecture that every natural number is the sum of m polygonal numbers of order m. Another chapter is devoted to indefinite ternary forms; the universal ones that represent all integers are found and so are the "zero forms" that represent the number zero in a non-trivial manner. The theorem giving the condition that the form $ax^2+by^2+cz^2$ be universal is especially noteworthy. Then there is a chapter on zero forms in four or more variables; it is proved that every indefinite form actually in five or more variables is a zero form. Finally there is a chapter on quadratic diophantine equations and a short one on the reduction of positive forms in n variables. Three chapters contain matters closely related to Waring's problem. In the first two of these we find the "nine cube theorem" with many generalizations to sums of cubes multiplied by coefficients and the representation of numbers by sums of "pyramidal numbers" and the like. The author's proof that every number is the sum of 37 perfect fourth powers is also given. Since Dirichlet's theorem on the primes in an arithmetical progression is used at various critical points in the book, a complete proof of this theorem is given in the Appendix. Much of the material here presented is the author's original work or that of his students; some of it is quite recent and various new results are given. The mathematical public is much indebted to Professor Dickson for putting these things into the form of a book. *H. W. Brinkmann* (Swarthmore, Pa.).

Stern, Erich. General formulas for the number of magic squares belonging to certain classes. Amer. Math. Monthly **46**, 555–581 (1939). [MF 676]

The author investigates and enumerates magic squares of prime order and of a certain type by replacing a magic square by two auxiliary latin squares. This is done writing each of the numbers from 0 to n^2-1 of the magic square as a two digit number in the n-ary scale and putting each digit in the corresponding cell of the two latin squares. It is shown that those squares, whose latin squares contain cyclically permuted elements in their rows, columns and main diagonals, constitute all known pandiagonal magic squares. The number of these squares is found to be $(n-3)(n-4)(n!)^2/8$. The author shows, however, that further pandiagonal squares exist. Non-pandiagonal squares are considered by combining a non-pandiagonal latin square with one which is or is not pandiagonal. In these respective cases there are produced $n(n-3)[(n-1)!]^2/2$ and $[(n-1)!]^2/4$ non-pandiagonal squares. Uniform step squares, symmetric squares, and symmetric uniform step squares are also enumerated. *D. H. Lehmer* (Bethlehem, Pa.).

van der Corput, J. G. Sur un certain système de congruences. II. Nederl. Akad. Wetensch., Proc. **42**, 707–712 (1939). [MF 419]

The proof of the theorems 1–3 of the first part [these Rev. 1, 39 (1940)] is concluded. *K. Mahler.*

Beeger, N. G. W. H. On the congruence $2^{p-1}\equiv 1 \pmod{p^2}$ and Fermat's last theorem. Nieuw Arch. Wiskde **20**, 51–54 (1939). [MF 567]

The author gives a table of Fermat's quotient $q_2=(2^{p-1}-1)/p$ modulo p for all primes between 14000 and 16000. There is also given for each such p the residue index of 2. This table is an extension of the author's table to 14000 [Messenger of Math. **51**, 149–50; **55**, 17–26] and shows that except for the primes 1093 and 3511, q_2 is not divisible by p for $p<16000$. By Wieferich's criterion, the first case of Fermat's last theorem is proved for exponents up to 16000. *D. H. Lehmer* (Bethlehem, Pa.).

Constantinescu, G. G. Integral solutions of the equation $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j=0$. Bol. Mat. **12**, 231–236 (1939). (Spanish) [MF 695]

A formula in $\frac{1}{2}n(n-1)$ parameters, giving integral solutions in terms of a given one, is derived by geometrical considerations. *G. Pall* (Montreal, Que.).

Gloden, A. Sur une méthode de résolution d'équations diophantiennes homogènes du troisième degré. Mathesis **53**, 233–235 (1939). [MF 855]

Beeger, N. G. W. H. Report on some calculations of prime numbers. Nieuw Arch. Wiskde **20**, 48–50 (1939). [MF 566]

This report is in three parts, the first of which gives an account of the construction of a factor table between the limits 999 999 000 and 1000 119 120. A complete list of errata in Poletti's list of primes between 10^9 and 10^9+10^6 [Tavole di Numeri Primi, Milan, 1920, 101–118] is given. The second part gives the factorizations of the numerators of the 14th, 15th, 16th and 17th Bernoulli numbers.

The third part presents some interesting data, prepared by Poletti, on the number of primes of the form x^2+x+A

for x not exceeding $k\cdot 10^3$ ($k=1, 2, \cdots, 11$) and for $A=41$, 19421, 27941, 72491. The quadratic functions corresponding to the last two values of A are exceedingly rich in primes and appear to be superior, in this respect, to the functions with $A=41$ and 19421. *D. H. Lehmer* (Bethlehem, Pa.).

van der Corput, J. G. Sur quelques fonctions arithmétiques élémentaires. Nederl. Akad. Wetensch., Proc. **42**, 859–866 (1939). [MF 702]

Let $f(n)$ be a multiplicative arithmetical function and $k>0$ be an integer such that the product

$$\prod_p \left(1+\frac{|g(p)|}{p}+\frac{|g(p^2)|}{p^2}+\cdots\right)$$

extended over the primes converges, where

$$g(p^\rho)=f(p^\rho)-\binom{k}{1}f(p^{\rho-1})+\binom{k}{2}f(p^{\rho-2})-\cdots+(-1)^\rho\binom{k}{\rho}f(1).$$

The author proves

$$(1)\qquad x^{-1}(\log x)^{1-k}\sum_{n\leq x}f(n)\to\frac{P}{(k-1)!},$$

where P denotes the product

$$P=\prod_p\left\{\left(1-\frac{1}{p}\right)^k\left(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\right)\right\}.$$

Let $\tau_r(n)$ be the number of the representations of n as a product of r positive integers. Then it follows from (1) that for every positive integer l

$$(2)\qquad x^{-1}(\log x)^{1-r^l}\sum_{n\leq x}\tau_r{}^l(n)\to\frac{P}{(r^l-1)!},$$

with

$$P=\prod_p\left\{\left(1-\frac{1}{p}\right)^r\left(1+\frac{\binom{r}{1}^l}{p}+\frac{\binom{r+1}{2}^l}{p^2}+\cdots\right)\right\}.$$

From (2) it follows, for $r=3$ and $l=2$, that $\sum_{n\leq x}\tau_3{}^2(n)$ has the order of magnitude $x(\log x)^8$, whereas Vinogradow [Bull. Acad. Sci. URSS, Cl. Sci. Math. Nat. **1938**, 399–416] obtained $x(\log x)^5$ erroneously.

A. Brauer (Princeton, N. J.).

van der Corput, J. G. et Pisot, Ch. Sur la discrépance modulo un. II. Nederl. Akad. Wetensch., Proc. **42**, 554–565 (1939). [MF 312]

Let for real α and λ the number of those elements u_ν of a finite system U of real numbers $u_1, u_2, \cdots, u_n$, for which

$$0\leqq u_\nu-\alpha-[u_\nu-\alpha]<\lambda-[\lambda],$$

be denoted by $N(\alpha, \alpha+\lambda)$ and let

$$N(\alpha, \alpha+\lambda)=(\lambda-[\lambda]+D(\alpha, \alpha+\lambda))n.$$

Then the upper bound $D(U)$ of $|D(\alpha, \alpha+\lambda)|$ for all α and λ is called the discrepancy of U. In the first part of the paper [same vol., 476–486] the authors proved

$$(1)\qquad D(U)<2^{(7/2)+(1/4\epsilon)}(D(U^*))^{1-\epsilon},$$

for any $\epsilon>0$, where U^* is the system of numbers $u_\nu-u_\rho$ ($\nu=1, \cdots, n$; $\rho=1, \cdots, n$). In this second part of the paper the authors deduce from (1): Let

$$\varphi(y)=\alpha\frac{y^k}{k!}+\alpha_1y^{k-1}+\cdots+\alpha_k$$

be a polynomial of degree $k\geqq 1$ and real coefficients α, and let

$$\left|\alpha-\frac{a}{q}\right|\leqq\frac{\tau}{q^2},$$

where a/q is an irreducible fraction with $q>0$ and where $\tau\geqq 1$. Then the discrepancy $D(Y)$ of the system $\varphi(1), \varphi(2), \cdots, \varphi(X)$ (where X is an integer not less than 3) satisfies for any $\epsilon>0$ the inequality

$$D(Y)<Cx^\omega\xi^{(1-\epsilon)2^{1-k}},$$

where $x=\log X$, $\xi=(\tau+qX^{-1})(q^{-1}+X^{1-k})$ and where C is a constant and ω depends only on k and ϵ. Since, as the authors prove too:

$$\left|n^{-1}\sum_{\nu=1}^n e^{2\pi i u_\nu}\right|\leqq 2\pi D(U),$$

this theorem gives non-trivial approximations for the Weyl-sums. *H. D. Kloosterman* (Leiden).

van der Corput, J. G. et Pisot, Ch. Sur la discrépance modulo un. III. Nederl. Akad. Wetensch., Proc. **42**, 713–722 (1939). [MF 420]

The authors prove: Let U be a system of n real numbers $u_1, u_2, \cdots, u_n$ and let V_l for any integer l with $0\leqq l<n$ be the system of numbers $u_{l+1}-u_1, u_{l+2}-u_2, \cdots, u_n-u_{n-l}$. Then

$$D(U)\leqq\frac{2l}{n}+2^\alpha(\omega)^{1/2},$$

where

$$\alpha=\frac{7}{2}+\left(\frac{|\log\omega|}{\log 2}\right)^{1/2},$$

$$\omega=\frac{1}{l+1}\left(1+\frac{1}{n}\right)+2\frac{D(V_1)+\cdots+D(V_l)}{l+1}+\frac{2l}{n}.$$

Here $D(U), D(V_1), \cdots, D(V_l)$ are the discrepancies of the systems $U, V_1, \cdots, V_l$, respectively [for the definition of this notion, cf. the reference on the second part of the paper under the same title]. Furthermore, the authors prove: If U is the system of numbers $f(1), f(2), \cdots, f(n)$, where $f(y)$ is a real function whose kth difference $\Delta^kf(y)$ is not less than $r>0$ for $y=1, 2, \cdots, n-k$ (where k is an integer less than n), then

$$D(U)<C\left\{(\rho^{-2}r)^{-1/(K-1)}+(rn^k)^{-2/K}+(\rho^{-1}rn)^{-2/K}\log\frac{1}{\rho}\right\}.$$

Here C is a constant and

$$\rho=\frac{1}{n-k}(\Delta^{k-1}f(n-k+1)-\Delta^{k-1}f(1)),\qquad K=2^k.$$

H. D. Kloosterman (Leiden).

van der Corput, J. G. Sur un lemme de M. Vinogradow. Nederl. Akad. Wetensch., Proc. **42**, 867–871 (1939). [MF 703]

The author considers the sum

$$S_m=\sum_{x,y}\psi(x)\omega(y)e(mf(xy)),\qquad e(u)=e^{2\pi iu},$$

where m is a positive integer, $f(u)=\frac{1}{2}\alpha u^2+\beta u$ (α, β real),

$$\left|\alpha-\frac{a}{q}\right|\leqq\frac{\tau}{q^2},\quad a, q\text{ integers};\ q>0;\ (a, q)=1;\ \tau\geqq 1,$$

and the summation is over a certain set E of lattice points (x, y). It is proved that, if $\psi(x)$ and $\omega(y)$ are defined for $|x| \leqq A$, $|y| \leqq B$ $(A, B \geqq 2)$ and satisfy

$$\sum_{|x| \leqq A} |\psi(x)|^2 \leqq A\Psi^2, \quad \sum_{|y| \leqq B} |\omega(y)|^2 \leqq B\Omega^2,$$

and if E lies in the rectangle $|x| \leqq A$, $|y| \leqq B$ and is such that for each x the range of y is a set of consecutive integers, then

$$|S_m| \leqq c_1 AB\Psi\Omega\left(\frac{1}{A^{1/4}} + (\log AB)^{(3l-1)/4l}\xi_m{}^{(l-1)/4l}\right),$$

where l is any integer not less than 2, c_1 depends only on l, and

$$\xi_m = \left(m\tau + \frac{q}{B}\right)\left(\frac{1}{q} + \frac{1}{A^2B}\right).$$

Vinogradow considered the case in which $\psi(x) = 1$, $\tau = 1$, and E is defined by $Y_0 < y \leqq Y_1$, $0 < xy \leqq N$ $(1 \leqq Y_0 < Y_1 \leqq N$, $N \geqq 3)$. The proof is similar in principle in the general case, and amounts in effect to estimating the "maximum" of a bilinear form by the inequality

$$\Big|\sum_{x,y} a_{x,y} u_x v_y\Big|^4 \leqq \Big(\sum_x |u_x|^2\Big)^2 \Big(\sum_y |v_y|^2\Big)^2 \sum_{x,x',y,y'} a_{x,y}\bar{a}_{x',y}\bar{a}_{x,y'}a_{x',y'},$$

where x and y range over the intervals $|x| \leqq A$ and $|y| \leqq B$, $u_x = \psi(x)$, $v_y = \omega(y)$, and $a_{x,y}$ is $e(mf(xy))$, when (x, y) belongs to E, and zero otherwise. The last sum is estimated by summing first for y, y', and using the conditions on E.

A. E. Ingham (Berkeley, Calif.).

Hasse, H. Simultane Approximation algebraischer Zahlen durch algebraische Zahlen. Nachr. Ges. Wiss. Göttingen. Fachgruppe I, N. F. **1**, 209–212 (1939). [MF 602]

Hasse, H. Simultane Approximation algebraischer Zahlen durch algebraische Zahlen. Monatsh. Math. Phys. **48**, 205–225 (1939). [MF 603]

The theorem of Thue-Siegel [Math. Z. **10**, 173–213 (1921)] for the approximation of an algebraic number α by the numbers ξ of a fixed algebraic number field K can be expressed as follows. Let $r > 1$ be the relative degree of α with regard to K. Let $f(x) = 0$ be the irreducible equation for ξ in the rational number field, written with integral, relatively prime coefficients, and denote by $H(\xi)$ the maximum of the absolute values of these coefficients. If we have $|\alpha - \xi| \leqq C/H(\xi)^e$ for an infinite sequence of primitive numbers ξ of K, where the positive constants C and e are independent of ξ, then

$$e \leqq \min_{s=0,1,\cdots,r-1} \{s + r/(s+1)\}.$$

Siegel's paper: Über einige Anwendungen Diophantischer Approximationen [Abh. Preuss. Akad. Wiss. Phys. Math. Kl. **1929**, 1–70] contains implicitly a generalization of this theorem for the approximation of a system of algebraic numbers. The author formulates this generalization and gives a new proof for it in the following form, which is sharper than the result contained in Siegel's paper: Let $\alpha_1, \alpha_2, \cdots, \alpha_g$ be a fixed system of algebraic numbers and K an algebraic number field of degree k. Denote by r the relative degree of $K(\alpha_1, \alpha_2, \cdots, \alpha_g)$ over K and denote by d the smallest integer such that the $(g+d)!/g!d!$ power products of $\alpha_1, \alpha_2, \cdots, \alpha_g$ of degrees $0, 1, \cdots, d$ are linearly dependent with regard to K. Let $\xi_i^{(\kappa)}$ $(\kappa = 1, 2, \cdots, k)$ be the conjugates of ξ_i $(i = 1, 2, \cdots, g)$, $N(1, \xi_1, \cdots, \xi_g)$ the norm of the ideal $(1, \xi_1, \cdots, \xi_g)$ in K, and

$$\begin{aligned} H &= H(\xi_1, \xi_2, \cdots, \xi_g) \\ &= \Big\{\prod_{\kappa=0}^{k-1} \max(1, |\xi_1^{(\kappa)}|, \cdots, |\xi_g^{(\kappa)}|)\Big\}/N(1, \xi_1, \cdots, \xi_g). \end{aligned}$$

If we have

$$|\alpha_i - \xi_i| \leqq C/H^e, \qquad i = 1, 2, \cdots, g,$$

for an infinite sequence of different systems $\xi_1, \xi_2, \cdots, \xi_g$ of K, where $C > 0$ and $e > 0$ are independent of $\xi_1, \xi_2, \cdots, \xi_g$, then

$$e \leqq \min_{\delta=0,1,\cdots,d-1} \{\delta + (C_{g+\delta,g})^{-1} r\}.$$

For $g = 1$, one obtains the theorem of Thue-Siegel and a new proof for this theorem. Moreover it follows that the theorem of Thue-Siegel is also true for the approximation by imprimitive numbers ξ and that it is true for $r = 1$. It may be mentioned that the latter fact was also proved by the reviewer [J. Reine Angew. Math. **160**, 70–99 (1929)] as a special case of a theorem on the approximation of an algebraic number by algebraic numbers of a fixed degree.

A. Brauer (Princeton, N. J.).

★**Hancock, Harris. Development of the Minkowski Geometry of Numbers.** The Macmillan Company, New York, 1939. xxiv+839 pp. $12.00.

In this book the author gives a faithful account of all the work of Minkowski on the geometry of numbers [Gesammelte Abhandlungen, 1911; Geometrie der Zahlen, 1910; Diophantische Approximationen, 1907] in the form of a translation and commentary. He does not connect Minkowski's results and methods with newer researches on the subject; and many simplifications and extensions of the theory, obtained during the last 25 years, are not mentioned. The book will be useful to those who are unable to read German or find Minkowski's own writings too difficult. It is well printed and contains a large number of excellent figures. Contents: (1) Preliminary notions (including Minkowski's inequality $M^3J \leqq 8$ for 3 dimensions, and some applications). (2) Surfaces that are nowhere concave. (3) The volume of bodies. (4) Bodies which with respect to their volumes have more than one point with integral coordinates. (5) Applications. (6) Algebraic numbers. (7) Arithmetical theory of a pair of lines, etc. (The chapters 2–6 correspond to Kapitel 1–4 in Geometrie der Zahlen.) (8) Shorter papers of Minkowski (historical remarks on quadratic forms, discriminant theorem, etc.). (9) A criterion for algebraic numbers. (10) The theory of continued fractions (containing among other things Blichfeldt's proof of the Tchebycheff-Minkowski theorem, and recent work of Dr. Pepper on an algorithm of Minkowski). (11) Periodic approximation of algebraic numbers. (12) On the approximation of a real quantity through rational numbers. (13) A further analytic-arithmetic inequality. (14) The arithmetic of the ellipsoid. (15) Computation of a volume through successive integrations. (16) Proof of the new analytic-arithmetic inequality. (17) The extreme standard bodies. (The chapters 13–17 correspond to Kapitel 5 of the Geometrie der Zahlen.) (18) Densest placement of congruent homologous bodies. (19) Miscellany. (20) New theory of quadratic forms, etc. (Minkowski's reduction theory; his asymptotic formula for the mean value of the class number of quadratic forms in n variables).

K. Mahler.

Bullig, G. Zur Zahlengeometrie in den total reellen kubischen Körpern. Math. Z. **45**, 511–532 (1939). [MF 406]

Let K_1, K_2, K_3 be three conjugate real cubic fields, $\mathfrak{R}$ a module of rank 3 of vectors $\mathfrak{m}=(\mu_1, \mu_2, \mu_3)$ with μ_i in K_i. This vector is called a minimum if no element $\mathfrak{x}=(\xi_1, \xi_2, \xi_3)\neq(0, 0, 0)$ of $\mathfrak{R}$ exists for which $|\xi_i|<|\mu_i|$ $(i=1, 2, 3)$. Let $\mathfrak{M}_{\mathfrak{R}}$ denote the set of all minima $\mathfrak{m}$ in $\mathfrak{R}$ with third coordinate $\mu_3>0$. To every minimum $\mathfrak{m}$ two "successors" $\mathfrak{m}_1$, $\mathfrak{m}_2$ are defined, which also belong to $\mathfrak{M}_{\mathfrak{R}}$. Author represents the elements of $\mathfrak{M}_{\mathfrak{R}}$ by points in a space with the same coordinates, and connects each point with its two successors by a line; a certain line complex $\mathfrak{M}^1$ is obtained in this way. The topological properties of this complex are studied, and it is proved that it can be laid into the plane such that its vertices, sides and polygons cover the plane completely and without overlapping. By a permutation of the coordinate axes, five similar complexes arise, but these can be derived from $\mathfrak{M}^1$. The paper depends on an earlier one by the author in Abh. Math. Sem. Hans. Univ. **12**, 369–414 (1938).

K. Mahler (Manchester).

Nagell, Trygve. Bestimmung des Grades gewisser relativalgebraischer Zahlen. Monatsh. Math. Phys. **48**, 61–74 (1939). [MF 626]

The author begins by giving a simple proof of a theorem due to Capelli [Rend. Accad. Sci. Fis. Mat. Napoli, **1897–1898**] which states a condition for the irreducibility of a binomial x^n-a. He then goes on to discuss, by elementary methods, the degrees of certain numbers determined by binomial equations. The following are typical results: (1) Let a, b be numbers in a field Ω such that $\alpha=a^{1/n}$, $\beta=b^{1/n}$ are of degree n over Ω. Then $\Omega(\alpha)$ and $\Omega(\beta)$ are identical if and only if $\alpha=c\cdot\beta^r$, where r is a natural number and c is in Ω. (2) Let n be an odd prime and α, β have the same meaning as in (1). Then the number $\alpha+\beta$ is of degree n^2 if and only if $\Omega(\alpha)$ and $\Omega(\beta)$ are not conjugate or identical.

H. W. Brinkmann (Swarthmore, Pa.).

Blichfeldt, H. F. Note on the minimum value of the discriminant of an algebraic field. Monatsh. Math. Phys. **48**, 531–533 (1939). [MF 667]

It is known that the absolute value of the discriminant of an algebraic field of degree n is greater than $((\pi/4)e^2)^n$ if n is sufficiently large, and that this lower bound can be replaced by e^{2n} in case the field is totally real [Minkowski, Geometrie der Zahlen, 1910, p. 134]. The author improves the latter result by showing that the discriminant of a totally real field is greater than $(2\pi e^{3/2})^n$ for large n. This is accomplished by using one of the author's theorems on lattice points [Trans. Amer. Math. Soc. **15**, 228, 230 (1914)].

H. W. Brinkmann (Swarthmore, Pa.).

Vandiver, H. S. On basis systems for groups of ideal classes in a properly irregular cyclotomic field. Proc. Nat. Acad. Sci. U.S.A. **25**, 586–591 (1939). [MF 450]

If l is an odd prime, the field k generated by a primitive lth root of unity is properly irregular if the class number H of k is divisible by l but the "second factor" of H is not. Let $H=q\cdot l^h$, where q is prime to l; the classes containing the qth powers of all ideals in k form a group of order l^h called the irregular class group. The author has shown elsewhere that the invariants of this group can be determined by examining the first $\frac{1}{2}(l-3)$ Bernoulli numbers [Proc. Nat. Acad. Sci. U. S. A. **15**, 202–207 (1929); Monatsh. Math. Phys. (1939)]. In this paper he gives explicitly a method for constructing systems of bases for this group. The author was led to this result by the examination of numerical data furnished by the extensive computations that he has carried out in connection with Fermat's "Last Theorem." From the form of these bases one can conclude, in particular, that there exists a class of ideals C such that any class in the irregular class group may be expressed as a power product formed from C and its conjugates.

H. W. Brinkmann (Swarthmore, Pa.).

Vandiver, H. S. On the composition of the group of ideal classes in a properly irregular cyclotomic field. Monatsh. Math. Phys. **48**, 369–380 (1939). [MF 653]

Let K be the field generated by a primitive lth root of unity (l an odd prime); if H is the class number of K and $H=q\cdot l^h$ (q not divisible by l), then the classes containing the qth powers of all ideals in K form a group of order l^h called the irregular class group. The main object of this paper is to determine the invariants of this group in case the field is properly irregular, that is, the "second factor" of H is not divisible by l. If e of the first $\frac{1}{2}(l-3)$ Bernoulli numbers are divisible by l, then there are e invariants l^{h_i}, $i=1, 2, \cdots, e$. The exponents h_i can also be determined by examining certain Bernoulli numbers. [This result was stated without proof in Proc. Nat. Acad. Sci. U. S. A. **15**, 202–207 (1929).] Additional properties of certain bases of the group are given. In the paper reviewed above the author gives an explicit method for constructing systems of such bases.

H. W. Brinkmann (Swarthmore, Pa.).

★**DeLury, D. B. On the representation of numbers by the indefinite form $ax^2+by^2+cz^2+dt^2$.** University of Toronto Studies, Mathematical Series, no. 5. University of Toronto Press, Toronto, 1938. 17 pp.

The purpose of this paper is to establish an asymptotic formula, for large k, for $R(m, k)$, the number of integral solutions of

$$ax^2+by^2+cz^2-dt^2=m, \qquad ax^2+by^2+cz^2+dt^2\leqq k,$$

where a, b, c, d, k are positive integers and m is integral. When $m\neq 0$, $R(m, k)$ is found to be either zero or $\sim(\pi/(\Delta)^{\frac{1}{2}})S(m)k$ as $k\to 0$, where $\Delta=abcd$ and $S(m)$ is Hardy's singular series [see Kloosterman, Acta Math. **49**, 406–464 (1926)]. Necessary and sufficient conditions that $R(m, k)=0$ are found. The methods extend to all indefinite forms $ax^2+by^2+cz^2+dt^2$ for $m\neq 0$ and to forms without cross products in five or more variables for all m. The methods are those Kloosterman [ibid.] used for the corresponding positive form.

B. W. Jones (Ithaca, N. Y.).

Ross, Arnold E. A theorem on simultaneous representation of primes and its corollaries. Bull. Amer. Math. Soc. **45**, 899–906 (1939). [MF 779]

Necessary and sufficient conditions that a ternary quadratic form

$$f=ax^2+by^2+cz^2+2ryz+2sxz+2txy$$

should represent all integers were given by Dickson [Studies in the Theory of Numbers, Chicago, 1930, 21–22] for the case $r=s=t=0$, and by the present author [Quart. J. Math. **4**, 147–158 (1933)] for the general case. In this paper a theorem on the simultaneous representation of primes by a form f and its reciprocal F is established and used to obtain the necessary conditions in the general case directly from the conditions in the special case $r=s=t=0$.

R. D. James (Saskatoon, Sask.).

Fuchs, W. H. J. and Wright, E. M. The 'easier' Waring problem. Quart. J. Math., Oxford Ser. **10**, 190–209 (1939). [MF 599]

Let $v(k)$ be the least s such that every integer n can be expressed in the form

$$F=x_1^k+x_2^k+\cdots+x_r^k-x_{r+1}^k-\cdots-x_s^k$$

for some r, $0\leqq r\leqq s$. E. M. Wright [J. London Math. Soc. **9**, 267–272 (1934)] proved that $v(2)=3$, $4\leqq v(3)\leqq 5$, $8\leqq v(4)\leqq 12$, $5\leqq v(5)\leqq 10$. Main object of this article is to find bounds for $v(k)$, $6\leqq k\leqq 20$. Define $\Delta(k, m, n)$, $\Gamma(k, m, n)$, $\Lambda(k, m, n)$, respectively, as the least s such that $F\equiv n \pmod m$ is soluble for some r, for $r=s$, for every r; $\Delta(k, m)=\max \Delta(k, m, n)$ for all n, and similarly for $\Gamma(k, m)$ and $\Lambda(k, m)$; $\Delta(k)=\max \Delta(k, m)$, etc. Hence $v(k)\geqq \Delta(k)$; and $v(k)\leqq 2j+\Delta(k, C)$ if we can find an identity

$$(x+a_1)^k+\cdots+(x+a_j)^k-(x+b_1)^k-\cdots-(x+b_j)^k=Cx+D.$$

The investigation of $\Delta(k)$ is in part like that of $\Gamma(k)$ by Hardy and Littlewood [Partitio Numerorum, IV and VIII; Math. Z. **12**, 161–188 (1922); Proc. London Math. Soc. (2) **28**, 518–542 (1928)]. Let $p^\Theta \| k$, $\phi=\phi_p=\Theta+1$ if $p>2$, $\phi_2=\Theta+2$, $\epsilon=(p-1, p^{-\Theta}k)$, $d=(p-1)/\epsilon$; let $\delta_p(k)$, $\gamma_p(k)$, $\lambda_p(k)$ be the least s for which $F\equiv n \pmod{p^{\phi_p}}$ is soluble in relative primes $x_1, \cdots, x_s$, for every n and (respectively) some r, $r=s$, every r. The problem reduces to moduli p^{ϕ_p}. Theorem 1: If k is odd, $\delta_2(k)=\lambda_2(k)=\gamma_2(k)=2$; if k is even, $\delta_2(k)=2^{\Theta+1}$, $\lambda_2(k)=2^{\Theta+2}$. Theorem 2: If k is odd, $\Delta(k)=\Lambda(k)=\Gamma(k)$. Theorem 3: If $p>2$, $d=1$, then $\lambda_p(k)=\gamma_p(k)=p^\phi$, $\delta_p(k)=\frac{1}{2}(p^\phi-1)$; if $d=2$, $\lambda_p(k)=\gamma_p(k)=\delta_p(k)=\frac{1}{2}(p^\phi-1)$. Theorem 4: If $p>2, \Theta=0, d>1$, then $\lambda_p(k)\leqq \max(\epsilon, 3)\leqq k$. Theorem 5: If $p>2, \Theta>0, d>2$, then $\lambda_p(k)\leqq p^\Theta\epsilon\leqq k$. Theorem 6: I. If $k=2^\theta$ $(\theta>1)$, $\Delta(k)=2^{\theta+1}$. II. If $k=\frac{1}{2}\pi^\theta(\pi-1)$, π an odd prime, $\theta>0$, then $\Delta(k)=\frac{1}{2}(\pi^{\theta+1}-1)\geqq k+1$. III. For k not in classes I or II, $\Delta(k)\leqq k$. IV. If $k=\frac{1}{2}(\pi-1)$, π an odd prime, k not in class I or II, $\Delta(k)=k$. Then $\Delta(k)$ is calculated for $k\leqq 36$. Identities from several sources are used to obtain upper bounds for $v(k)\leqq 2j+\Delta(k, C_k)$; for $k=6, \cdots, 20$ these bounds are 14, 14, 30, 29, 32, 28, 38, 39, 53, 69, 98, 72, 107, 115, 133. For large k the best upper bound is $v(k)\leqq G(k)+1$, with Vinogradow's result for $G(k)$. *G. Pall.*

van der Corput, J. G. et Pisot, Ch. Sur un problème de Waring généralisé. III. Nederl. Akad. Wetensch., Proc. **42**, 566–572 (1939). [MF 313]

Conditions are stated for the existence of solutions y_ν of the system

$$t_\mu=\sum_{\nu=1}^{n} b_{\mu\nu}f_\nu(y_\nu), \qquad \mu=1, \cdots, m,$$

$$y_\nu\equiv a_\nu \pmod{A_\nu}, \quad |f_\nu(y_\nu)|\leqq X, \nu=1, \cdots, n,$$

where some or all of the y_ν are primes. The numbers t_μ, $b_{\mu\nu}$, a_ν, A_ν, and X are given integers satisfying certain conditions and $f_\nu(y_\nu)$ are polynomials of given degree k with integral coefficients. In some cases an approximate formula for the number of representations can be given. The results are said to follow from a general theorem of van der Corput [Acta Arith. **3**, 181–234 (1939)] and recent contributions of Siegel, Walfisz and Vinogradov. *R. D. James.*

Lehmer, D. H. On the remainders and convergence of the series for the partition function. Trans. Amer. Math. Soc. **46**, 362–373 (1939). [MF 469]

The author writes the Hardy-Ramanujan formula and the Rademacher formula for $p(n)$ in the following forms, respectively:

$$(1) \quad p(n)=\frac{12^{1/2}}{24n-1}\sum_{k=1}^{N} A_k^*(n)\left(1-\frac{k}{\mu}\right)e^{\mu/k}+R_1(n, N),$$

$$(2) \quad p(n)=$$
$$\frac{12^{1/2}}{24n-1}\sum_{k=1}^{N} A_k^*(n)\left\{\left(1-\frac{k}{\mu}\right)e^{\mu/k}+\left(1+\frac{k}{\mu}\right)e^{-\mu/k}\right\}+R_2(n, N),$$

with $\mu=\mu(n)=(\pi/6)(24n-1)^{1/2}$. Hardy and Ramanujan proved that, for every fixed $\alpha>0$,

$$(3) \qquad R_1(n, \alpha n^{1/2})=O(n^{-1/4}),$$

whereas Rademacher gave an estimate of $R_2(n, N)$ explicitly depending on n and N which has as consequence on one hand $\lim_{N\to\infty} R_2(n, N)=0$ and on the other hand the relation (3). Of basic importance in the discussion of the remainders $R_1(n, N)$ and $R_2(n, N)$ is an estimate of the "Kloosterman sum" $A_k(n)=k^{1/2}A_k^*(n)$. Trivial is only $|A_k(n)|\leqq k$. In a previous paper the author [Trans. Amer. Math. Soc. **43**, 271–295 (1938)] had completely evaluated the $A_k(n)$ by means of the factorization of k and derived the estimate $|A_k(n)|<2k^{5/6}$. Nevertheless, the question whether for certain fixed n $A_k(n)$ was bounded or not for $k\to\infty$ was not completely settled. This time the author proves (Theorems 5 and 6) that for any given positive or negative n there exist infinitely many primes p such that $|A_p(n)|>3^{1/2}p^{1/2}$. This implies the divergence of the series (1). More precisely, Theorem 7 states that for every positive n the kth term in (1) is, for infinitely many values of k, greater in absolute value than $13k(24n-1)^{-3/2}$. In numerical applications the convergent Rademacher series seems to show a very rapid approximation of $p(n)$. This is, however, only apparent and is due to the fact that only a limited precision is required since $p(n)$ is known to be an integer. The series as such is not better convergent than $\sum k^{-2}$. In fact the author proves (Theorem 8): For every positive n there exist infinitely many k such that the kth term of the series (2) is in absolute value greater than $(43/34)k^{-2}$. Two further questions are treated in this paper. Firstly, the author gives estimates of $R_1(n, N)$ and $R_2(n, N)$ which are better than the previously known ones. These improvements are useful for numerical applications. Secondly, since the Hardy-Ramanujan paper the choice $N=[\alpha n^{1/2}]$ has been of particular interest as far as the order of magnitude of the remainders is concerned. The author improves (3) and obtains here a sort of final result, namely

$$R_i(n, \alpha n^{1/2})=O(n^{-1/2}\log n), \qquad i=1, 2.$$

H. Rademacher (Swarthmore, Pa.).

Rankin, R. A. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. I. The zeros of the function $\sum_{n=1}^{\infty}\tau(n)/n^s$ on the line $\Re s=13/2$. II. The order of the Fourier coefficients of integral modular forms. Proc. Cambridge Philos. Soc. **35**, 351–372 (1939). [MF 540, 541]

Ramanujan's function $\tau(n)$, defined by

$$\sum_{n=1}^{\infty}\tau(n)q^{2n}=q^2\{(1-q^2)(1-q^4)\cdots\}^{24}=\Delta(\tau), \quad q=e^{\pi i\tau}, \Im\tau>0,$$

possesses multiplicative properties embodied in the formula

$$g(s)=\sum_{n=1}^{\infty}\tau(n)n^{-s}=\prod_p(1-\tau(p)p^{-s}+p^{11-2s}), \; s=\sigma+it, \; \sigma>13/2,$$

and $g(s)$, which can be continued as an entire function satisfying the functional equation

$$(2\pi)^{-s}\Gamma(s)g(s)=(2\pi)^{s-12}\Gamma(12-s)g(12-s),$$

has no zeros in $\sigma>13/2$. It is proved here, in paper I, that $g(s)$ has no zeros on $\sigma=13/2$. The proof is based on the inequality

$$|4\cos A\cos B|\leqq 2+\cos 2A+\cos 2B,\quad \cos A,\cos B \text{ real},$$

(an extension of the classical inequality with $B=0$), and on such properties of the function

$$f(s)=\sum_{n=1}^{\infty}\tau^2(n)n^{-s},\qquad \sigma>12,$$

as can be derived from the multiplicative properties of $\tau(n)$ and from Hardy's inequalities

$$(1)\quad A_1n^{12}\leqq\tau^2(1)+\tau^2(2)+\cdots+\tau^2(n)\leqq A_2n^{12},\quad 0<A_1<A_2.$$

In paper II the inequalities (1) are replaced by an asymptotic formula. More generally, it is proved that, if

$$H(\tau)=\sum_{n=1}^{\infty}a(n)e^{2\pi in\tau/N},\qquad \tau=x+iy,\ y>0,$$

is an integral modular form of dimensions $-\kappa<0$ and Stufe N, which vanishes at all rational cusps of the fundamental region and is absolutely convergent for $y>0$, then

$$(2)\qquad \sum_{n\leqq x}|a(n)|^2=\alpha x^{\kappa}+O(x^{\kappa-2/5}),\qquad \alpha>0.$$

This implies $a(n)=O(n^{\frac{1}{2}\kappa-1/5})$, an improvement on the $O(n^{\frac{1}{2}\kappa-\frac{1}{4}+\epsilon})$ of Salié and Davenport, and a further step in the direction of Ramanujan's conjecture $|\tau(n)|\leqq n^{11/2}d(n)=O(n^{11/2+\epsilon})$, since $\tau(n)$ is an instance of $a(n)$ with $\kappa=12$, $N=1$. The proof is based on a study of the function

$$f(s)=\sum_{n=1}^{\infty}|a(n)|^2n^{-s},\qquad \sigma>\kappa.$$

When $N=1$ (the simplest case), the author starts from the formula

$$(4\pi)^{-s}\Gamma(s)f(s)=\iint_S y^{s-1}|H(\tau)|^2dxdy,$$

where S is $(|x|\leqq\frac{1}{2},\ y\geqq 0)$, subdivides S into fundamental regions, and transforms each region into the particular region $D(|x|\leqq\frac{1}{2},\ |\tau|\geqq 1)$ by a modular substitution. Summation over fundamental regions and multiplication by $2\zeta(2s-2\kappa+2)$ introduce under the integral sign a factor $\xi(s,\tau)=\sum\sum'|m\tau+n|^{-2s+2\kappa-2}$ (summed over all integers m, n, except $m=n=0$), which can be studied by classical methods (as an Epstein zeta-function). The final result is that $f(s)$ can be continued as a meromorphic function satisfying the equation $\varphi(s)=\varphi(2\kappa-1-s)$, where

$$\varphi(s)=(2\pi)^{-2s}\Gamma(s)\Gamma(s-\kappa+1)\zeta(2s-2\kappa+2)f(s),$$

and $\varphi(s)$ is regular at all finite points except for simple poles at $s=\kappa$ and $s=\kappa-1$. An asymptotic formula

$$\sum_{n\leqq x}|a(n)|^2\sim\alpha x^{\kappa}$$

follows at once from the Wiener-Ikehara theorem, and the more precise relation (2) is obtained by applying a general theorem of Landau to the Dirichlet's series for $\zeta(2s)f(s+\kappa-1)$. The case $N>1$ is similar in principle but more complicated formally. *A. E. Ingham.*

Speiser, Andreas. Die Funktionalgleichung der Dirichletschen L-Funktionen. Monatsh. Math. Phys. **48**, 240–244 (1939). [MF 642]

The author first proves certain facts concerning a determinant involving the representations of finite Abelian groups. This is a generalization, from the case of cyclic groups, of the results used by I. Schur [Nachr. Ges. Wiss. Göttingen **1921**, 147–153] in his evaluation of the Gaussian sums. These results enable the author to prove the functional equation for the Dirichlet L-functions with the use of a relatively small amount of analysis.

H. S. Zuckerman (Seattle, Wash.).

Watson, G. N. Über Eigenschaften des Ramanujanschen Kettenbruches. Monatsh. Math. Phys. **48**, 516–530 (1939). [MF 666]

The author proves certain functional equations for Ramanujan's continued fraction

$$R(q)=\frac{q^{1/5}|}{|1}+\frac{q|}{|1}+\frac{q^2|}{|1}+\frac{q^3|}{|1}+\cdots,\qquad |q|<1.$$

Using the notation $R(q^n)\equiv R_n$, $R(q)\equiv R_1\equiv R$, $\bar{R}\equiv -R(e^{\frac{1}{5}\pi i}q)$, the relations are:

$$(1)\quad (\bar{R}-R)(1+\bar{R}_5R_5)=\bar{R}R(\bar{R}_4+R_4)-5R^2\bar{R}^2(\bar{R}-R)^2,$$

$$(2)\quad R_2-R^2=RR_2^2(R_2+R^2),$$

$$(3)\quad (R_3-R^3)(1+RR_3^3)=3R^2R_3^2,$$

$$(4)\quad (R^5+R_4^5)(1-RR_4)=RR_4(1+R^4R_4^4)-5R^2R_4^2(RR_4-1)^2,$$

$$(5)\quad R^5=R_5\frac{1-2R_5+4R_5^2-3R_5^3+R_5^4}{1+3R_5+4R_5^2+2R_5^3+R_5^4},$$

$$(6)\quad R'+R_4=(1-R'R_4)\left(\frac{\sqrt{5}-1}{2}\right),\quad R(q')\equiv R',$$

$$\log\frac{1}{q}\log\frac{1}{q'}=\pi^2.$$

$$(7)\quad S'+S_{4/5}=(1-S'S_{4/5})\left(\frac{\sqrt{5}-1}{2}\right)^2(\sqrt{5}-2),\qquad R'^5\equiv S'.$$

Five others of a more complex structure are also given. Most of these twelve relationships were first given, without proof, by Ramanujan in his celebrated notebooks. The author gave a proof of (5) in J. London Math. Soc. **4**, 29–48 (1929). Proof of (3) is reserved for another paper. Proofs of the others are given here for the first time. The proof involves the application of numerous properties of the theta functions, the modular equations and to a certain extent of the Rogers-Ramanujan identities. The author also indicates that a relation between R_n and R exists for all positive integers n; such a relationship may be obtained through the use of the modular equation corresponding to the transformation of order n of the elliptic functions. It follows that a relation between R_m and R_n exists for all integers m, $n>0$. *M. A. Basoco* (Lincoln, Neb.).

ANALYSIS

✱McLachlan, N. W. Complex Variable & Operational Calculus with Technical Applications. Cambridge University Press, Cambridge, 1939. xi+355 pp. \$6.50.

The book is intended as a modern treatment of operational calculus for engineers and mathematicians in industry. It consists of four parts: theory of complex variable (chap. 1–7), theory of operational calculus (chap. 8–10), technical applications (chap. 11–15), and appendices con-

taining additional theory for the first two parts and an extensive list of references.

In the first part the treatment of contour integration and the calculus of residues is given with a minimum of introductory theory. The examples are selected from the applications and theory of operational calculus. In view of the purpose of the book the plan here seems commendable. The reader is given no practical tests for determining where a given function is analytic, however; the Cauchy-Riemann conditions are not mentioned. This question of analyticity, which arises in nearly every application, is left to the reader's intuition. Some confusion arises from the occasional use of infinitesimal gaps in contours, of infinite semi-circles, and failure to distinguish between the cases of finite and infinite sets of poles. In evaluating the Bromwich-Wagner integrals on pages 67–71, for example, where the integrands have an infinite number of poles, it is not proved that the integrals are equal to the series obtained; neither is the convergence of the series or the integrals established. [Complete solutions of such problems are illustrated by the reviewer in Math. Z. 43, 743–748 (1938).]

Although the theory of the operational calculus here is based on the Laplace integral transformation and the Mellin inversion theorem the argument in the proof of the latter is not presented in full, and the conditions of validity are narrower and more numerous than they need be. No attempt is made in the book to use these conditions to fully establish an inversion or interpretation of the operational form; still nearly every interpretation here, simple or involved, is made by means of the integral in the inversion theorem for the sake of practice in using the complex variable. The important rule for the operational form of the derivative of a function whose initial value is not zero is obtained on page 124, but it is almost ignored afterwards. Again, very little use is made of the Borel (Faltung) theorem.

The part on applications reverts to the old system of using one symbol for both the function and its operational form. Although it is not stated, the initial values of functions are assumed to be zero. Since the problems in partial differential equations are not completely stated as boundary value problems these applications seem involved and uncertain. In fact the solution of the heat problem on page 259 is incorrect because of the failure to note the initial condition; the solution given is for an initial temperature of $-10°$ throughout instead of $20°$. Only formal solutions are made to the problems in partial differential equations. The selection of physical problems is interesting and refreshing; so are the devices for obtaining approximate solutions. The book seems unusually free from typographical errors.

R. V. Churchill (Ann Arbor, Mich.).

Niven, Ivan. The transcendence of π. Amer. Math. Monthly 46, 469–471 (1939). [MF 358]

This paper repeats the classical proof as to be found in Perron's Irrationalzahlen, page 178 or Landau's Zahlentheorie II, page 93, but without avoiding integrals.

A trifling slip: On page 471, second formula from below, the factor τ^{p+1} is omitted. *P. Scherk.*

Scardina, Mariano. Alcune proposizioni sulle disuguaglianze. Riv. Fis. Mat. Sci. Nat. 13, 516–520 (1939). [MF 559]

Elementary remarks on inequalities between a^x and b^x, if $a>b$ and $ab<0$. *W. Feller* (Providence, R. I.).

Northcott, D. G. Some inequalities between periodic functions and their derivatives. J. London Math. Soc. 14, 198–202 (1939). [MF 243]

Let $f(x)$ have the period 2π, $f^{(k)}(x)$ be absolutely continuous, $\int_0^{2\pi} f(x)dx=0$. By proper integral representations, the author shows that

$$\max |f(x)| \leqq \alpha_k \cdot \max |f^{(k)}(x)|,$$

where

$$\alpha_k = 4/\pi \sum_{\nu=0}^{\infty} (-1)^{\nu(k+1)}(2\nu+1)^{-k-1}.$$

This constant can not be replaced by a smaller one.

G. Szegö (Stanford University, Calif.).

Popoviciu, Tiberiu. Deux remarques sur les fonctions convexes. Bull. Sect. Sci. Acad. Roum. 20 (1938), 187–191 (or 45–49) (1939). [MF 617]

Popoviciu, Tiberiu. Sur l'approximation des fonctions convexes d'ordre supérieur. Bull. Sect. Sci. Acad. Roum. 20 (1938), 192–195 (or 50–53) (1939). [MF 618]

Popoviciu, Tiberiu. Sur le prolongement des fonctions monotones et des fonctions convexes définies sur un nombre fini de points. Bull. Sect. Sci. Acad. Roum. 20 (1938), 196–198 (or 54–56) (1939). [MF 619]

The contents of these three notes are described in the following three paragraphs.

1. Let E be a linear set, bounded and closed. Let the real function $f(x)$ $(x\epsilon E)$ be continuous in E. It is furthermore assumed that, whenever $f(x)+\alpha x$ (α=constant) assumes its minimum value in two points x_1, x_2 of E, it assumes this minimum value for every point of E between x_1 and x_2. It is shown that these assumptions imply that $f(x)$ is non-concave in E, that is, the familiar inequality condition is verified for any three points of E. A second result concerns the following property of a non-concave function $f(x)$: The set E may be decomposed in two "consecutive" subsets E_1 and E_2 (one of which may be vacuous) such that $f(x)$ is non-increasing in E_1 and non-decreasing in E_2. This property, however, does not characterize non-concave functions. Likewise, the inequality

$$(1)\quad f(x_2) < \max [f(x_1), f(x_3)], \qquad \text{if } x_1<x_2<x_3;\ x_1, x_2, x_3 \epsilon E,$$

is verified by non-concave functions, but does not characterize them. It is shown that the three-point condition (1) characterizes precisely those functions which enjoy the first mentioned property concerning monotoneity on consecutive subsets of E.

2. The author continues his investigations on continuous functions which are non-concave of order n [see Mathematica 8, 1–85 (1934); 10, 49–54 (1935)], that is, having nonnegative divided differences of order n. It is shown that a function with these properties in a finite interval (a, b) can be there uniformly and indefinitely approximated by polynomials which are convex of order n on the whole real axis. Bernstein's polynomials are used in the proof.

3. Let $f(x)$ be defined on the set of m points $x_1<x_2<\cdots<x_m$. The following two theorems are established. I. If $f(x_1)<f(x_2)<\cdots<f(x_m)$, then there exist polynomials $P(x)$ increasing on the whole real axis such that $P(x_i)=f(x_i)$ $(i=1, \cdots, m)$. II. If $f(x)$ is convex on the given set of m points, then there exist polynomials $P(x)$ convex on the whole real axis such that $P(x_i)=f(x_i)$.

I. J. Schoenberg (Waterville, Me.).

John, F. Special solutions of certain difference equations. Acta Math. **71**, 175–189 (1939). [MF 200]

Solutions of a difference equation like (i) $f(x+1)-f(x)=g(x)$, $x>0$, may be characterized, apart from prescribing values for a unit interval, by their asymptotic behaviour [cf. N. E. Nörlund's Hauptlösungen] or by local properties. Two instances of the latter kind are quoted: I. In (i) let $g(x)=\log x$. Then every continuous convex $f(x)=\text{const.}+\log\Gamma(x)$ [for instance, Courant, Differential and Integral Calculus, vol. 2]. II. The unique convex solution of (ii) $1/f(x+1)=xf(x)$ is $F(x)=(\frac{1}{2})^{\frac{1}{2}}\Gamma(\frac{1}{2}x)/\Gamma(\frac{1}{2}x+\frac{1}{2})$ [A. E. Mayer, Acta Math. **70** (1938)]. The author first generalizes I and II. Notation: cc. means continuous+convex, $\uparrow$ non-decreasing, $\downarrow$ non-increasing; $\sum$ to be taken from $n=1$ to ∞, lim for $x\to\infty$. Theorem A: If greatest lower bound $g(x)=0$, every two $\uparrow$ solutions $f(x)$ of (i) differ only by a constant. It is sufficient for the existence of a $\uparrow f(x)$ that $g(x)\downarrow$ to 0; then $f(x)=\text{const.}-g(x)+\sum(g(n)-g(x+n))$. By application, II results (with cc. instead of convex). Theorem B: If $\liminf g(x)/x=0$ every two cc. $f(x)$ differ by a constant. Sufficient condition: $g'(x)$ continuously $\downarrow$ to 0; then $f(x)=f(1)-x\,\text{const.}-g(x)-\sum(g(x+n)-g(n)-xg'(n))$. Special case: I, with the addition of discontinuous convex solutions; A and B are substantially comprised in the general theorem on the equation $a_0f(x+n)+a_1f(x+n-1)+\cdots+a_nf(x)=g(x)$. Let all roots of $a_0\rho^n+a_1\rho^{n-1}+\cdots+a_n=0$ be simple and of absolute value 1. If $g'(x)$ is continuous and $\lim g'(x)=0$, every two cc. $f(x)$ differ by a constant; so do every two $\uparrow f(x)$ if $\rho=1$ is a root and $\lim g(x)=0$; if $\rho=1$ is not a root and (a) $\lim g(x)/x=0$ or (b) $\lim g'(x)/x=0$ then there exists at most one (a) $\uparrow f(x)$ or (b) cc. $f(x)$. The proof is based on the difference of two $f(x)$ being almost periodic. It is stated that under suitable restrictions for $g(x)$ the $f(x)$ obtained above are identical with the Hauptlösungen.

A. E. Mayer (London).

Menger, Karl. On Cauchy's integral theorem in the real plane. Proc. Nat. Acad. Sci. U. S. A. **25**, 621–625 (1939). [MF 722]

The author gives a geometric condition which is both necessary and sufficient in order that the integral $\int pdx+qdy$, where p and q are continuous in a rectangle R, have the same value for any two co-terminal (rectifiable) curves in R. The condition is in the form of the existence of a sequence of rectangular nets of a certain sort covering R whose norms and quotients tend towards zero. The norm of a net is defined in terms of the lengths of the subdivisions while the quotient involves the functions p and q under consideration. The fact that the condition can be given in terms of a single sequence of nets throws light on the classical arguments. An alternative sufficient condition is obtained in which the nets have a somewhat simpler character. This latter condition treats certain cases when p and q are nowhere differentiable.

W. T. Martin (Cambridge, Mass.).

Fulton, Dawson G. Further generalizations of the Cauchy integral formula. Amer. J. Math. **61**, 843–852 (1939). [MF 282]

Cauchy's integral formula

$$f(a)=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{z-a}$$

may be written, on separating real and imaginary parts, in the form

$$2\pi X_0{}^1=\int_C\frac{1}{r^2}\{X^1(x^1\alpha^1+x^2\alpha^2)+X^2(x^1\alpha^2-x^2\alpha^1)\}ds,$$

$$2\pi X_0{}^2=\int_C\frac{1}{r^2}\{X^1(x^2\alpha^1-x^1\alpha^2)+X^2(x^1\alpha^1+x^2\alpha^2)\}ds,$$

where $r^2=x^2+y^2$, $a=0$, $f(z)=X^2+iX^1$, $z=x^1+ix^2$, $dz=dx+idy=i(\alpha^1+i\alpha^2)ds$, α^1 and α^2 being the direction cosines of the normal to the contour. The author investigates under what conditions integrals of the form

$$\int_C\frac{1}{r^2}N^{i\nu}_{\sigma\lambda}x^\sigma X^\lambda\alpha_\nu ds,\qquad i,\nu,\sigma,\lambda=1,2,$$

where $N^{i\nu}_{\sigma\lambda}$ are arbitrary constants and $r^2=g_{\alpha\beta}x^\alpha x^\beta$ is a positive definite quadratic form, will represent a pair of functions at any interior point a of a region R in terms of their values on C, the boundary of R. The author obtains a sufficient condition which, aside from questions of differentiability, involves a system of differential equations analogous to the Cauchy-Riemann equations as well as certain algebraic relations between the constants $g_{\alpha\beta}$ and $N^{i\nu}_{\sigma\lambda}$. An analogous result is stated for n-dimensional regions.

W. Seidel (Rochester, N. Y.).

Calculus

Baidaff, B. I. Derivatives of functions of u/v. Bol. Mat. **12**, 170–176 (1939). (Spanish) [MF 867]

Opatowski, I. Combinatoric interpretation of a formula for the nth derivative of a function of a function. Bull. Amer. Math. Soc. **45**, 944 (1939). [MF 787]

Shohat, J. A. Definite integrals and Riemann sums. Amer. Math. Monthly **46**, 538–545 (1939). [MF 674]

Based on the idea of equal distribution, the author calculates certain definite integrals by direct approximation by Riemann sums. A semiconvergent expansion of the remainder in Simpson's formula is also obtained. *G. Szegö.*

Arley, Niels. On a Dirichlet integral. Mat. Tidsskr. B **1939**, 49–51 (1939). (Danish) [MF 715]

By an inductive argument it is shown that, provided $\lambda_i\neq\lambda_k$ for $i\neq k$,

$$\int_{t_1+\cdots+t_n=t}dt_1\cdots\int dt_{n-1}\exp\left[-(\lambda_1t_1+\cdots+\lambda_nt_n)\right]$$
$$=\sum_{k=1}^n\frac{e^{-\lambda_kt}}{(\lambda_k-\lambda_1)\cdots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})\cdots(\lambda_k-\lambda_n)}.$$

This formula was needed for a problem of radioactivity. In mathematical language, one is concerned with a simple stochastic process which plays a role in various fields. It was dealt with also by the reviewer [these Rev. 1, 22 (1940)].

W. Feller (Providence, R. I.).

Raynor, G. E. On Serret's integral formula. Bull. Amer. Math. Soc. **45**, 911–917 (1939). [MF 781]

Serret's formula reads

$$\int_0^{\pi/2}\cos^p\varphi\sin q\varphi d\varphi$$
$$=\frac{\Gamma(p+1)}{\Gamma\left(1+\frac{p+q}{2}\right)\Gamma\left(1+\frac{p-q}{2}\right)}\int_0^1\frac{t^{p-q/2}-t^{p+q/2}}{(1-t)(1+t)^{p+1}}dt,$$

where $\Re(p) > -1$. The author deduces from it several other definite integrals. He states that the formula is misprinted in the Enzyklopädie Math. Wiss. 2, 2, 853.
W. Feller (Providence, R. I.).

Tricomi, Francesco. Dimostrazione della formula di Stirling relativa ad $n!$, per via del tutto elementare. Atti Accad. Sci. Torino **74**, 105–109 (1939). [MF 487]

The proof given here is essentially the same as, for example, that given in Courant, Differential and Integral Calculus, vol. 1, chap. 7. *W. Feller* (Providence, R. I.).

Turrière, Émile. Une nouvelle courbe de transition pour les raccordements progressifs: la radioïde pseudo-elliptique. Bull. Soc. Math. France **67**, 62–99 (1939). [MF 446]

The author considers the problem met by railway and highway engineers of constructing transition curves which join a straight section to a circular section of road without giving discontinuities or undue variation of the curvature. To the collection of such curves already used, called radioides, including the cubic parabola, the lemniscate of Bernoulli, Cornu's spiral, and others, the author adds a new curve, the pseudo-elliptic radioide, $\tan (x/m) = \tanh (y/m)$. Among the advantages proposed for this curve is the fact that all data needed for its construction, such as length, curvature, etc., are easily calculated in terms of elementary transcendental functions without the use of the elliptic integrals and higher transcendentals which appear in connection with other radioides. The properties of this new transition curve are worked out in the paper with great detail and accuracy. *J. W. Green* (Rochester, N. Y.).

Egger, Hans. Über besondere Seilkurven. Z. Angew. Math. Mech. **19**, 319–320 (1939). [MF 493]

The shape of a curve with homogeneous mass distribution under the influence of central forces is given by means of explicit integrals. *W. Feller* (Providence, R. I.).

Pedrazzini, Pierino. Sulla trattrice ordinaria e sulle curve di inseguimento. Period. Mat. **19**, 146–149 (1939). [MF 563]

Fourier Series and Integrals, Theory of Approximation

Favard, J. Remarque sur les polynomes trigonométriques. C. R. Acad. Sci. Paris **209**, 746–748 (1939). [MF 842]

The author considers a real trigonometric polynomial

$$P_n(x) = \tfrac{1}{2} \sum_{-n}^{n} c_k e^{i\lambda_k x}$$

$(\lambda_0 = 0,\ \lambda_k = -\lambda_{-k},\ \lambda_k < \lambda_{k+1},\ c_k = \bar{c}_{-k})$, with $\delta = \min (\lambda_{k+1} - \lambda_k) > 0$, and shows that for all numbers a and l,

$$\text{(1)} \qquad \sum_{-n}^{n} |c_k|^2 \geqq K(l,\delta) \int_a^{a+l} |P_n(x)|^2 dx.$$

The method is the elementary one used by Marcinkiewicz and Zygmund [Duke Math. J. 4, 469–472 (1938)] to establish a theorem of N. Wiener ((1) with the inequality reversed). The inequality (1) (with $P_n(x)$ not required to be real, but with a different K) has been established previously by Paley and Wiener [Fourier Transforms in the Complex Domain, 1934, p. 123]. *R. P. Boas, Jr.* (Durham, N. C.).

Carafa, Mario. Applicazione delle serie di Fourier all'inversione delle funzioni e al calcolo numerico delle radici reali di una equazione. Rend. Sem. Mat. Roma **3**, 73–83 (1939). [MF 502]

Let $f(x)$ be continuous and monotone in (a, b) and have a derivative; let $\varphi(x)$ be a given function and $P(u)$ a function such that $\varphi(x) = P[f(x)]$. It is shown first that, in the case $f(a) = 0$, $f(b) = 2\pi$,

$$P(f) = (2\pi)^{-1} \int_a^b \varphi(x) f'(x) dx + \pi^{-1} \sum_{k=1}^{\infty} (a_k \cos kf + b_k \sin kf),$$

$$a_k = \int_a^b \varphi(x) \cos kf(x) \cdot f'(x) dx,$$

$$b_k = \int_a^b \varphi(x) \sin kf(x) \cdot f'(x) dx.$$

The author studies convergence properties of this series under various additional assumptions concerning derivability of f and φ. In the case $\varphi(x) = x$ the formula above gives an expansion of the function inverse to $f(x)$ and may be also used for computation of roots of the equation $f(x) = 0$. It also may be used for expansion of implicit functions. Finally the author indicates some extensions to the case of functions of two variables. *J. D. Tamarkin.*

Salem, R. Sur les transformations des séries de Fourier. Fund. Math. **33**, 108–114 (1939). [MF 832]

Let $\Omega(n)$ be positive, increasing and concave and $\Omega(n) \to \infty$ as $n \to \infty$. Let

$$\tfrac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be the Fourier series of a function $f(x)$. The author proves that if f is continuous then $\Omega(n)$ can always be determined so that

$$\tfrac{1}{2} a_0 \Omega(0) + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) \Omega(n)$$

is also the Fourier series of a continuous function. The theorem remains correct if the word "continuous" is replaced by "integrable." *J. D. Tamarkin.*

Dubourdieu, M. J. Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes. Compositio Math. **7**, 96–111 (1939). [MF 370]

The results of this paper coincide with the results of the first part of W. Feller's very recent paper: Completely monotone functions and sequences [Duke Math. J. **5** (1939); these Rev. **1**, 52 (1940)]. The author gives a proof of S. Bernstein's theorem on completely monotone functions and establishes independently the inversion formula given by Feller. As the author's publication is somewhat prior to Feller's, it may be referred to as the Dubourdieu-Feller inversion formula. As compared to Feller's proof, the author's procedure is somewhat longer but very lucid and elementary. The connections with the calculus of probabilities are stressed. Thus Lemme 1 states in substance that if a random variable x is distributed according to Poisson's law of parameter λ ($\lambda = \mu x$ in the paper), then the distribution of the variable $(x - \lambda)/\sqrt{2\lambda}$ tends to the normal law as $\lambda \to \infty$. This is a very special case of a general limit theorem and is readily established by means of characteristic functions. The author prefers to give an elementary proof along classical lines. *I. J. Schoenberg* (Waterville, Me.).

Goodspeed, F. M. Some generalizations of a formula of Ramanujan. Quart. J. Math., Oxford Ser. **10**, 210–218 (1939). [MF 600]

A formula concerning Fourier cosine transforms, stated by Ramanujan and proved under definite conditions by Hardy [Quart. J. Math., Oxford Ser. **8**, 245–254 (1937)], is replaced by the following more general and more symmetrical result: If $\chi(u)$ is an integral function such that

$$\chi(\sigma+it)/\{2^{\frac{1}{2}|\sigma|}\Gamma(\tfrac{1}{2}|\sigma|+\tfrac{1}{2}it)\}=O(e^{A|\sigma|+B|t|}),\quad B<\pi,$$

for all real σ and t, then the series

$$\sum_{0}^{\infty}\frac{(-1)^n\chi(n)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})}x^n,\quad \sum_{0}^{\infty}\frac{(-1)^n\chi(-n-1)}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+\frac{1}{2})}y^n$$

converge for $0\leqq x<e^{-A}$ and $0\leqq y<e^{-A}$, and represent analytic functions $L(x)$ and $M(y)$ which are regular for all positive x and y and are Fourier cosine transforms of one another. The proof is by calculus of residues. An extension to general Watson transforms is given, and a special case related to a Hilbert transform is discussed.

A. E. Ingham (Berkeley, Calif.).

Gröbner, Wolfango. Sistemi di polinomi ortogonali soddisfacenti a date condizioni. Rend. Sem. Mat. Roma **3**, 29–53 (1939). [MF 509]

The object of this paper is the construction of a sequence $\{p_n(x)=x^n+\cdots\}$ of polynomials satisfying certain prescribed orthogonality conditions on a given interval (a,b) with respect to a given weight-function $Q(x)$; $p_n(x)$ further satisfies a set of linear homogeneous equations

$$(1)\qquad \sum_{j=0}^{2}\alpha_{ij}p_n^{(j)}(a)=\sum_{j=0}^{2}\beta_{ij}p_n^{(j)}(b),\qquad i=1,2,\cdots,s;$$

$$\alpha_{ij},\ \beta_{ij}\text{ given constants};\ p^{(0)}\binom{a}{b}\equiv p\binom{a}{b}.$$

The problem is reduced to minimizing, under conditions (1), the integral $\int_a^b p_n^2(x)Q(x)dx$, which is treated by methods of calculus of variations (Euler-Lagrange multipliers). As special cases of (1) the author first obtains known results concerning Legendre, Jacobi and Laguerre Polynomials. The author further treats new cases. To illustrate: $(a,b)\equiv(0,1)$, $Q(x)=1$, $p_n(0)=p_n(1)=0$, or $p_n(0)=0$ [cf. D. Jackson, Amer. Math. Monthly **46**, 493–497 (1939). Ref.]. Here the explicit expressions of a few first polynomials are given, also a recurrence relation, Darboux Formula. The paper closes with a discussion of the case where $\{p_n''(x)+p_n(x)\}$ is orthogonal on $(0,1)$, with $Q(x)=1$, under the condition $p_n(0)=p_n(1)=0$. *J. A. Shohat* (Philadelphia, Pa.).

Webster, M. S. Note on certain Lagrange interpolation polynomials. Bull. Amer. Math. Soc. **45**, 870–873 (1939). [MF 772]

Let $l_k^{(n)}(x)$, $1\leqq k\leqq n$, denote the fundamental polynomials of Lagrange interpolation associated with the roots of $\sin(n+1)\theta/\sin\theta$, $x=\cos\theta$. Then $|l_k^{(n)}(x)|<2$, $-1\leqq x\leqq +1$, and this constant can not be replaced by a smaller one. Similar results are established for the fundamental polynomials associated with the roots of $\sin(2n+1)(\theta/2)/\sin(\theta/2)$.

G. Szegö (Stanford University, Calif.).

Cassina, Ugo. Formole sommatorie e di quadratura con l'ordinata media. Atti Accad. Sci. Torino **74**, 300–325 (1939). [MF 513]

The author considers a summation formula, in terms of symmetric ordinates, of the following type:

$$(1)\qquad \sum_{r=0}^{2n}f_r=cf_n+\sum_{s=0}^{m-1}c_s(f_s+f_{2n-s})+R_f,$$

$$c,\ c_s\text{ constants};\ n>m-1,$$

where f_r is the customary notation for a given ordinate. We may write (1) in terms of successive differences

$$(2)\qquad S=Cf_n+\sum_{s=0}^{m-1}C_s[\Delta^sf_{2n-s}+(-1)^s\Delta^sf_0]+R.$$

If in (2) $f(x)$ is a polynomial of degree less than $2m+2$, the C_s satisfy a system of linear equations given explicitly and solved for $m=0,1,2,3,4$. For the remainder R in (2), the author obtains an expression in terms of Bernoulli polynomials, also of the form

$$R=2\theta\cdot\frac{(m!)^2}{(2m+2)!}\Delta_{2m+2}\binom{n-m+1}{2}\left(\frac{2n+1}{2m+1}\right)\cdot(-1)^m,\qquad 0<\theta<1,$$

where Δ_{2m+2} is a certain intermediate value of $\Delta_{2m+2}f(t)$, t varying from 0 to $2n-2m-1$. The author then turns to definite integrals. Expressing $\int_a^b f(x)dx$ as the limit of a Riemannian sum, he derives from (2) a corresponding mechanical quadratures formula

$$(3)\qquad \int_a^b f(x)dx=2^{2n}\frac{(m!)^2}{(2m+1)!}(b-a)f\left(\frac{a+b}{2}\right)+\sum_{s=0}^{m-1}(b-a)^{s+1}\cdot\gamma_s[D^sf(b)+(-1)^sD^sf(a)]+R',$$

and an expression for R' involving $D^{2m+2}f$. Explicit formulae (3) are given for $m=0,1$ (trapezoidal and Simpson's formulae, respectively), 2, 3, 4. Comparison is made with other mechanical quadratures formulae (Peano, Newton-Cotes).

J. A. Shohat (Philadelphia, Pa.).

✱Jordan, Charles. Calculus of Finite Differences. Hungarian Agent Eggenberger Book-Shop, Budapest, 1939. xxi+654 pp.

Book is designed for practical use by computers, statisticians and actuaries, and includes the following topics: (1) Operators for differences, displacements and means; symbolical calculus; generating functions; expansion of functions. (2) Factorials; gamma, digamma and trigamma functions; binomial coefficients; Beta-functions; exponential and logarithmic functions. (3) Summation. Indefinite sums; methods of determining sums. (4) Stirling's numbers of first and second kind. Applications to expansions of functions. Thiele's semi-invariants. (5) Bernoulli numbers and polynomials. Maclaurin-Euler summation formula. Gregory's summation formula. (6) Euler's and Boole's polynomials. Applications to expansion and summation. (7) Direct and inverse interpolation; Newton's, Gauss's, Bessel's, Stirling's, Everett's and Lagrange's formulas for interpolation; construction of tables. (8) Approximation by least squares, using Legendre polynomials, Hermite polynomials, Fourier series for the case of continuous variable and integration; orthogonal polynomials and trigonometric sums for the case of discrete variables and summation; graduation of data. (9) Numerical solution of equations; Newton's, Horner's and root-squaring methods; numerical integration; solution of differential equations. (10) Several independent variables. Double entry tables. (11) Difference equations. Homogeneous linear, constant coefficients; nonhomogeneous; variable coefficients. (12)

Partial difference equations. Methods of Laplace, Boole, Fourier, Lagrange, Ellis. *W. E. Milne* (Corvallis, Ore.).

Steffensen, J. F. Note on divided differences. Danske Vid. Selsk. Math.-Fys. Medd. **17**, no. 3, 12 pp. (1939). [MF 461]

The author obtains a formula for the rth divided difference of a product analogous to Leibnitz's formula for the rth derivative of a product. If $\phi(x_0x_1\cdots x_r)$ denotes the rth divided difference of $\phi(x)$ and if $\phi(x)=f(x)g(x)$, then

$$\phi(x_0\cdots x_r)=\sum_{\nu=0}^{r}f(x_0\cdots x_\nu)g(x_\nu\cdots x_r).$$

More generally, if $\phi(x)=f_1(x)f_2(x)\cdots f_n(x)$, then

$$\phi(x_0\cdots x_r)=\sum f_1(x_0\cdots x_\alpha)f_2(x_\alpha\cdots x_\beta)f_3(x_\beta\cdots x_\gamma)\cdots f_n(x_\rho\cdots x_r),$$

the summation extending to all values for which $0\leqq\alpha\leqq\beta\leqq\gamma\leqq\cdots\leqq\rho\leqq r$. Particular cases of above formulas for coincident arguments and for equidistant arguments are also given. By specialization of $f(x)$ and $g(x)$ a neat derivation is secured for Newton's interpolation formula with divided differences, the remainder term differing slightly from the usual form. *W. E. Milne* (Corvallis, Ore.).

Differential Equations

Mitrinovitch, D. S. Sur un problème de Darboux. Bull. Sect. Sci. Acad. Roum. **20** (1938), 135–137 (1939). [MF 615]

The author reduces the equation $dy/dx+y^2=hf^2(x)$ by the substitution $\xi=\int f(x)dx$, $\eta=yf-f'/2f$ to $d\eta/d\xi+\eta^2=h+\phi(\xi)$. This is related to a theorem of Darboux [Théorie des Surfaces, 1915, 2, p. 210, 2nd ed.]. *W. Feller.*

Zwirner, Giuseppe. Un'osservazione su un problema ai limiti per le equazioni differenziali. Boll. Un. Mat. Ital. **1**, 334–336 (1939). [MF 575]

Let $f(x, y, y')$ be a real valued function of the real variables (x, y, y') which is continuous in (y, y') for fixed x and measurable as a function of x for fixed (y, y') on S: $x_0\leqq x\leqq x_1$, $-\infty<y<+\infty$, $-\infty<y'<+\infty$; moreover, there exists a finite and non-negative integrable (summable) function $\psi(x)$ on $x_0\leqq x\leqq x_1$ such that $|f(x, y, y')|<\psi(x)$ on S. Under these hypotheses, the author shows by elementary approximation methods that for arbitrary values y_0, y_1 the equation

$$y(x)=\int_{x_0}^{x}dt\int_{x_0}^{t}f(u, y(u), y'(u))du+y_0+\frac{x-x_0}{x_1-x_0}\left[y_1-y_0-\int_{x_0}^{x_1}dt\int_{x_0}^{t}f(u, y(u), y'(u))du\right]$$

has at least one solution. *W. T. Reid* (Chicago, Ill.).

Hardy, G. H. A note on a differential equation. Proc. Cambridge Philos. Soc. **35**, 652–653 (1939). [MF 839]

This note proves the following theorem: If the equation $y'''-yy''+2(y'^2-1)=0$ has a solution $y(x)$ that is regular for large positive values of x, and if $y'\to1$ when $x\to\infty$, then $y(x)=x+b$, where b is a constant. The proof is accomplished through use of a lemma to the effect that the differential equation has no solution for which the quantity $y'-1$ is of fixed sign and approaches zero.

W. M. Whyburn (Los Angeles, Calif.).

Morant, J. Extension d'une remarque de M. Goursat sur le développement en série entière de l'intégrale d'une équation différentielle. Mathesis **53**, 214–215 (1939). [MF 851]

This paper considers the differential equation

$$d^2y/(dx)^2=F(x, y, y'),$$

where $F(x, y, y')$ is holomorphic in a neighborhood of $x=y=y'=0$. A procedure due to Goursat is used to obtain the Maclaurin expansion for the unique solution of this equation which satisfies $y(0)=y'(0)=0$. A sequence of linear second order differential equations is set up in such a way that the terms of the desired Maclaurin series are obtained from series solutions for these linear equations.

W. M. Whyburn (Los Angeles, Calif.).

Zwirner, Giuseppe. Su un problema di valori al contorno per equazioni differenziali del secondo ordine. Rend. Sem. Mat. Roma **3**, 57–70 (1939). [MF 511]

This paper is concerned with the determination of a solution of the second order differential equation $y''=f(x, y, y')$ satisfying the boundary conditions $y(a)=\alpha$, $y(b)=\beta$. The function $f(x, y, y')$ is defined in a region of the type C: $a\leqq x\leqq b$, $\sigma(x)\leqq y\leqq\tau(x)$, $-\infty<y'<+\infty$, and $\sigma(a)\leqq\alpha\leqq\tau(a)$, $\sigma(b)\leqq\beta\leqq\tau(b)$. The result obtained is less general than that previously proved by Scorza Dragoni [Rend. Sem. Mat. Roma (4) 2, 255–275 (1938); Atti Accad. Naz. Lincei. Rend. (6) 28, 317–325 (1938)], since in addition to the hypotheses on $f(x, y, y')$ imposed by Scorza Dragoni it is assumed in the present paper that $f(x, y, y')$ satisfies a generalized Lipschitz condition with respect to y, y' in every bounded portion of C. With the additional assumption, however, Zwirner solves the boundary value problem by elementary methods; in particular, he avoids the extension of the range of definition of $f(x, y, y')$ used by Scorza Dragoni in the above-mentioned papers.

W. T. Reid (Chicago, Ill.).

Li, Ta. Über die allgemeine lineare Differentialgleichung. Comment. Math. Helv. **12**, 1–19 (1939). [MF 496]

This paper develops formulas for the solutions of ordinary, homogeneous, linear differential equations of the nth order, where the coefficients are bounded functions of x and the equations are of non-singular type. The solutions are obtained in the form of infinite series with terms made up of multiple integrals which contain the coefficient functions and certain polynomials. These series are very similar to those obtained by the Picard method of successive approximations. Convergence of the series is established under various hypotheses on the coefficients. The special case $y^{(n)}-f(x)y=0$ is treated in detail. Other special cases considered are (a) the nth order equation with constant coefficients; (b) the nth order Cauchy equation, that is, with coefficient of $y^{(i)}$ equal to x^i; (c) the first order equation $y'-f(x)y=0$; (d) the equation $y^{(n)}-dy=0$, where d is a constant. In these special cases, the author shows that his expressions for the solutions yield the solutions as they occur in the classical treatments. *W. M. Whyburn.*

Kienast, Alfred. Bemerkungen zur vorstehenden Arbeit von Herrn Ta Li. Comment. Math. Helv. **12**, 20–24 (1939). [MF 497]

This note is designed to clear up certain questions that arise from the paper by Mr. Ta Li. Points in the preceding paper which were not carefully stated are here clearly formulated and rigorous proofs given to make them free

from question. Convergence of the series developed by Ta Li is established under the usual hypotheses, that is, that the coefficients be Lebesgue integrable and bounded. This note also treats a case where the equation is of singular type.

W. M. Whyburn (Los Angeles, Calif.).

Schmidt, Erhard. Bemerkung zum Fundamentalsatz der Theorie der Systeme linearer partieller Differentialgleichungen. I. Ordnung. Monatsh. Math. Phys. **48**, 426–432 (1939). [MF 659]

The fundamental theorem established in this paper is the following: Let operators $A(f)$, $B(f)$, and $K(f)$ be defined by the formulas

$$A(f)=\sum_{i=1}^{n}a_i\frac{\partial f}{\partial x_i},\quad B(f)=\sum_{i=1}^{n}b_i\frac{\partial f}{\partial x_i},$$

$$K(f)=\sum_{i=1}^{n}[A(b_i)-B(a_i)]\frac{\partial f}{\partial x_i},$$

where the functions a_i, b_i, and f are continuous together with their first partial derivatives with respect to their arguments $x_1, \cdots, x_n$. If the equations $A(f)=0$, $B(f)=0$ both hold in a neighborhood of a point then the equation $K(f)=0$ also holds. It is observed that this theorem follows directly from the identity $K(f)=A(B(f))-B(A(f))$ in case f is assumed to have second partial derivatives. In the proof of the present paper, second partial derivatives are not assumed to exist. The proof consists of first proving the lemma that, if W is a cube in n-dimensional space with edges parallel to the coordinate axes, $R(W)$ its boundary and N its outer normal, then the formulas

$$\int_{R(W)}F[G_{x_i}\cos(Nx_j)-G_{x_j}\cos(Nx_i)]d\sigma$$

$$=\int_W[F_{x_j}G_{x_i}-F_{x_i}G_{x_j}]d\tau$$

hold, where $F(x_1, \cdots, x_n)$ and $G(x_1, \cdots, x_n)$ are continuous together with their first partial derivatives. Application of this lemma yields

$$K(f)=\left[\sum_{i=1}^{n}\frac{\partial b_i}{\partial x_i}\right]A(f)-\left[\sum_{j=1}^{n}\frac{\partial a_j}{\partial x_j}\right]B(f)$$

$$+\lim_{h\to 0}\left[\frac{1}{h^n}\int_{R(W_h)}\{(\sum_{i=1}^{n}a_i\cos(Nx_i))B(f)$$

$$-(\sum_{i=1}^{n}b_i\cos(Nx_i))A(f)\}d\tau\right],$$

and the theorem follows at once from this formula for $K(f)$.

W. M. Whyburn (Los Angeles, Calif.).

Orloff, Constantin. Sur la formation de l'intégrale générale d'une équation aux dérivées partielles du second ordre, au moyen d'une intégrale complète. J. Math. Pures Appl. **18**, 145–156 (1939). [MF 380]

Being given an integral

$$z=z(x, y, C_1, C_2, C_3, C_4, C_5)$$

of a partial differential equation of second order with two independent variables depending on five arbitrary constants, the author considers the possibility of making a transformation on the constants which will reduce the integral to a certain special form, such that it is possible to eliminate three of the constants from the integral and its first derivatives with respect to x and y. Having obtained in this manner two first integrals, each depending on two arbitrary constants, following Lagrange the author uses variation of constants to obtain two first integrals, each containing an arbitrary function. He then obtains the general integral by quadratures. The method is illustrated with the equation for the vibrating string. The author also gives an example to show that it is sometimes possible to pass from a first integral with more than two arbitrary constants directly to a first integral with an arbitrary function.

E. W. Titt (Hyattsville, Md.).

Collatz, L. Genäherte Berechnung von Eigenwerten. Z. Angew. Math. Mech. **19**, 224–249 (1939). [MF 143]

Collatz, L. Genäherte Berechnung von Eigenwerten. Z. Angew. Math. Mech. **19**, 297–318 (1939). [MF 492]

This paper reports on various methods of approximate calculation of characteristic values, in particular on such methods which have proved successful in application to mechanics. The considered characteristic values are, in general, those of a differential equation $L[\varphi]+\lambda p\varphi$, where $p>0$ and L is a linear differential expression of the second order for functions φ of one or more variables s.

The successive approximations through $L[F_n]+pF_{n-1}=0$ lead to the expressions $\mu_{2n}=\int pF_nF_{n-1}ds/\int pF_n^2ds$ and $\mu_{2n+1}=\int pF_n^2ds/\int pF_{n+1}F_nds$, which decrease $(\mu_1\geqq\mu_2\geqq\mu_3\geqq\cdots)$ and are above the lowest characteristic value λ_1. For the latter statement $(\mu_m\geqq\lambda_1)$, which comprises Rayleigh's principle $(\mu_2\geqq\lambda_1)$, a continuity proof is given. An estimate of λ_1 from below can be obtained from (*) $\mu_{n+1}-(\lambda_1/\mu_{n+1})\leqq(\mu_n-\mu_{n+1})/(l_2-\mu_{n+1})$, provided a lower estimate $l_2>\mu_{n+1}$ for λ_2 is known.

In the next section the methods of Ritz, Galerkin, and Grammel are explained. The first two methods differ only as to the class of admitted functions. Ritz's method is derived from the minimum problem for $-\int\varphi L[\varphi]ds/\int p\varphi^2ds$; Grammel's method differs from it only in so far as it is derived from the minimum problem for $\int p^{-1}L[\varphi]^2ds/-\int\varphi L[\varphi]ds$. These methods give upper estimates. Trefftz and Willers have shown an interesting way of obtaining lower estimates, based on the fact that the sums $\sum_{i=1}^{\infty}\lambda_i^{-1}$ and $\sum_{i=1}^{\infty}\lambda_i^{-2}$ can easily be expressed in terms of the Green's function. The author then discusses the inclusion formula of Temple: $(-L[w]/pw)_{\min}\leqq\lambda_1\leqq(-L[w]/pw)_{\max}$ and that of D. H. Weinstein: $\mu_2-(\mu_1\mu_3-\mu_2^2)^{\frac{1}{2}}\leqq\lambda_l\leqq\mu_2+(\mu_1\mu_3-\mu_2)^{\frac{1}{2}}$, where λ_l is the characteristic value nearest to μ_2. The important methods of A. Weinstein, however, are not mentioned. A lower estimate of Trefftz and Newing is derived; it is, incidentally, more complicated though less accurate than that obtained from the above formula (*) for $n=1$.

The second part reports on perturbation methods, on Dunkerley's formula for combined systems, and on other related subjects. Finally, in extensive numerical calculations the discussed methods are applied to various examples and compared as to their efficiency.

K. O. Friedrichs.

Davis, Arthur W. Differentiability and continuity properties of solutions of certain partial differential equations of applied mathematics. Iowa State Coll. J. Sci. **14**, 20–21 (1939). [MF 742]

Abstract of a thesis.

Lowan, A. N. Correction to "On Green's functions in the theory of heat conduction in spherical coordinates." Bull. Amer. Math. Soc. **45**, 951–952 (1939). [MF 790]

Cf. Bull. Amer. Math. Soc. **45**, 407–413 (1939).

Carslaw, H. S. and Jaeger, J. C. A problem in conduction of heat. Proc. Cambridge Philos. Soc. 35, 394–404 (1939). [MF 545]

The problem of conduction of heat in a solid bounded internally by a sphere $r=a$ and consisting of a concentric layer of one material and a second layer, extending to infinity, of another material is dealt with. The surface $r=a$ is kept at a constant temperature and the initial temperature of the whole is zero. The solution is first obtained as a contour integral. Deforming the path of integration, the contour integral is transformed into a real one. Then it is proved that the solution in terms of a contour integral, which has been obtained by formally using Laplace transform, satisfies the differential equations and the boundary conditions. *A. Erdélyi* (Edinburgh).

Bateman, H. On some symmetrical potentials and the partial differential equation $V_{ssss}+V_{ssyy}+V_{tt}=0$. Monatsh. Math. Phys. 48, 322–328 (1939). [MF 649]

The author constructs special solutions of the equation mentioned in the title in the form of integrals involving Bessel functions. Thus, under suitable restrictions, the following expressions U, V, W are recognized as solutions:

$$U=\int_0^\infty \cos(kz)K_0(kw)u(s, kt, k)dk,$$

$$V=\int_0^\infty \sin(kz)K_0(kw)v(s, kt, k)dk,$$

$$W=\int_0^\infty e^{-ks}J_0(kw)f(s+kt, k)dk,$$

where f is arbitrary and $u(X, Y, k)$, $v(X, Y, k)$ are conjugate harmonics in X, Y; $(w^2=x^2+y^2)$. Special instances are given, where $U+V$ is a function of type W. Integrals of that type and relations between them have been discussed in connection with physical applications [L. V. King, Proc. Roy. Soc. London, ser. A. 139, 237–277 (1933); M. Muscat, Physics 4, 129–147 (1933)]. On expressing u and v in terms of their values for $Y=0$, a solution in the form

$$\int_0^\infty K_0(kw)dk\int_0^\infty e^{-hkt}[a(h, k)\cos(hs-kz) + b(h, k)\sin(hs-kz)]dh,$$

with arbitrary functions a, b, is obtained. *F. John.*

Sicardi, Francesco. Unicità della soluzione di un'equazione a derivate parziali del 4° ordine a caratteristiche multiple. Boll. Un. Mat. Ital. 1, 331–334 (1939). [MF 574]

The author considers the parabolic differential equation

$$\delta\delta z=\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial y}\right)z=\frac{\partial^4 z}{\partial x^4}-2\frac{\partial^3 z}{\partial x^2\partial y}+\frac{\partial^2 z}{\partial y^2}=0.$$

Let Γ be a region bounded from above by a characteristic line segment $y=$const. and from below by an arc s with continuously turning tangent, and intersected in at most one point by every parallel to the y-axis. He proves that a solution z is uniquely determined in Γ if z, $\partial z/\partial x$, $\partial z/\partial y$ are prescribed on s. The proof consists in a suitable transformation of $\iint_\Gamma(\partial z/\partial y)\delta\delta z dxdy$ with the help of Green's formula. *F. John* (Lexington, Ky.).

Calculus of Variations

✶Green, J. W. Aspects of the Calculus of Variations. Notes after lectures by Hans Lewy. University of California Press, Berkeley, Calif., 1939. vi+96 pp.

In these lectures little emphasis is placed on the usual subject matter of the elementary calculus of variations; the attention is turned instead to boundary value problems and the problem of Plateau. Chapter I contains the derivation of the first necessary conditions for a minimum. Chapter II is devoted to integral equations and boundary value problems. Chapter III consists of a very brief discussion of the second variation and sufficiency theorems. In Chapter IV an existence theorem is established for non-parametric problems in the plane; the method is that of Lewy [Math. Ann. 98 (1927)]. Chapter V is devoted to the study of harmonic functions and the Dirichlet problem. This prepares for the study of the problem of Plateau and conformal mapping in Chapter VI. The solution of the problem of Plateau here presented is that of Courant for the single-contour case. *E. J. McShane.*

Morse, Marston. Sur le calcul des variations. Ann. Inst. H. Poincaré 9, 1–11 (1939). [MF 377]

An expository outline of a recent form of the abstract theory of functional topology [developed in detail by the author in memoirs in Ann. of Math. 38, 386–449 (1937), and Mémor. Sci. Math. no. 92 (1939)] in terms of rank, homotopic critical point, accessibility and upper reducibility. Included is an example illustrating the convenience of using Vietoris cycles rather than singular cycles in the theory. *C. B. Tompkins* (Princeton, N. J.).

Tonelli, Leonida. Su alcuni funzionali. Ann. Mat. Pura Appl. 18, 1–21 (1939). [MF 581]

In preparation for a study of functionals of the form

$$\iint_Q F(x, z, y(x), y(z), y'(x), y'(z))dxdz,$$

Q being the unit square, the author discusses in this paper the existence of maxima and minima of the functionals

$$\mathfrak{J}(\varphi)=\iint_Q K(x, z)\varphi(x)\varphi(z)dxdz,$$

$$\mathfrak{J}_1(\varphi)=\mathfrak{J}(\varphi)+\int_0^1 H(x)\varphi(x)dx,$$

$$\mathfrak{J}_2(\varphi)=\iint_Q K(x, z)\varphi^2(x)\varphi^2(z)dxdz,$$

$$\mathfrak{J}_3(\varphi)=\mathfrak{J}_2(\varphi)-\left[\int_0^1 P(x)\varphi(x)dx\right]^\alpha, \qquad 0\leqq\alpha<4.$$

It is proved that, if $K(x, z)$ is a positive (negative) definite, or semi-definite kernel, then the functional $\mathfrak{J}(\varphi)$ has an absolute maximum (minimum) in the class C of real, single-valued functions φ of class L^2 on $(0, 1)$ for which $\int_0^1\varphi^2dx=1$. If the maximizing function φ_0 makes $\mathfrak{J}(\varphi_0)$ positive, then φ_0 is a characteristic function of $\frac{1}{2}[K(x, z)+K(z, x)]$ and $1/\mathfrak{J}(\varphi_0)$ is the corresponding characteristic value, indeed, it is the least positive characteristic value; conversely, if this kernel has positive characteristic values, then there is

a least among these and its reciprocal is the value of the absolute maximum of the functional $\mathfrak{J}(\varphi)$ in the class C. Hence a new proof is obtained of the classical result that, if $K(x, z)$ is a symmetric kernel, different from zero on a set of positive measure, then it has at least one real characteristic value. The functional $\mathfrak{J}_1(\varphi)$ is shown to have an absolute maximum in the class C, provided $H(x)$ belongs to class L^2 on $(0, 1)$ and K is a positive definite, or semi-definite kernel. Throughout the proofs, use is made of functionals of the same forms as those mentioned above, on a class of derivatives of absolutely continuous functions. Defining $\mathfrak{J}_2'(y)$ as $\mathfrak{J}_2(y')$, it is shown that $\mathfrak{J}_2'(y)$ is a lower semi-continuous functional on the class A' of real, single-valued absolutely continuous functions $y(x)$, for which $y(0)=0$ and y' belongs to class L^2 on $(0, 1)$, provided $K(x, z)$ is continuous and positive in the unit square (various ways are suggested in which the conditions on K can be weakened). This result is then used to show that the functional $\mathfrak{J}_3(\varphi)$ possesses an absolute minimum in the class of functions C', which is obtained from the class C by dropping the condition $\int_0^1\varphi^2dx=1$, provided the function $P(x)$ belongs to L^2 on $(0, 1)$. *A. Dresden.*

McShane, E. J. On multipliers for Lagrange problems. Amer. J. Math. **61**, 809–819 (1939). [MF 278]

The author shows that along every minimizing curve for a variational problem of the Lagrange or Bolza type there exist corresponding multipliers $[\lambda_0, \lambda_1(x), \cdots, \lambda_m(x)]$ with $\lambda_0 \geqq 0$ for which the multiplier rule, the necessary condition of Weierstrass and their usual consequences all hold. This result goes beyond those previously established in that the author makes no assumption whatsoever of normality. It is deduced from the properties, in particular the convexity, of a certain set defined by the first variation of the problem. *W. T. Reid* (Chicago, Ill.).

Douglas, Jesse. Solution of the inverse problem of the calculus of variations. Proc. Nat. Acad. Sci. U. S. A. **25**, 631–637 (1939). [MF 725]

The author announces the complete solution of the inverse problem of the calculus of variations: to determine whether a given family of curves $y_i=y_i(x)$ in $(n+1)$-space, consisting of the solutions of a system of second-order differential equations, can be identified with the totality of extremals of a problem $\int\varphi dx=\min$, and, if the identification is possible, to find all the corresponding functions φ. The analogous parametric problem is discussed in tensor form. The details are to appear in a later paper. *E. J. McShane* (Charlottesville, Va.).

Hölder, Ernst. Die infinitesimalen Berührungstransformationen der Variationsrechnung. Jber. Deutsch. Math. Verein. **49**, 162–178 (1939). [MF 686]

This paper gives an exposition of the connection between the path curves of 1-parameter groups of contact transformations and the extremals of problems of the calculus of variations, involving simple or multiple integrals, along the general lines indicated by Vessiot [Bull. Soc. Math. France **34**, 230 (1906); **40**, 68 (1912); Ann. École Norm. **26**, 405 (1909)]. The work closely parallels the treatment given by Carathéodory in his "Variationsrechnung," for simple integrals, and in Acta Litt. Sci. Szeged **4** (1928) for multiple integrals [compare also Weyl, Ann. of Math. **36**, 607 (1935), and Boerner, Math. Ann. **112**, 187 (1936)]. The special contribution made in this paper consists in the construction of 1-parameter groups of contact transformations which generate the transversal manifolds which play a central role in Caratheodory's theory, and in pointing out the relation between the characteristic function of the infinitesimal transformations of this group, and the integrand function of the related variations problem. *A. Dresden* (Swarthmore, Pa.).

Sakellariou, Nilos. Zur Variationsrechnung. Monatsh. Math. Phys. **48**, 314–321 (1939). [MF 648]

Let

$$\delta\int\int F(p_{ik})dudv=0,\quad p_{ik}=\frac{\partial(x_i, x_k)}{\partial(u, v)},\quad p_{ik}=-p_{ki}, \tag{1}$$

be a regular variational problem in parametric form in which F is of class C'', and homogeneous of the first degree in the p_{ik}. We suppose also that F satisfies the differential equations

$$\frac{\partial F}{\partial p_{ik}}\frac{\partial F}{\partial p_{\lambda\rho}}+\frac{\partial F}{\partial p_{i\rho}}\frac{\partial F}{\partial p_{k\lambda}}+\frac{\partial F}{\partial p_{i\lambda}}\frac{\partial F}{\partial p_{\rho k}}=0,$$

$$i, k, \lambda, \rho=1, \cdots, n;\quad \frac{\partial F}{\partial p_{ik}}=-\frac{\partial F}{\partial p_{ki}};\quad i>k.$$

Let $x=x_i(u, v)$ be an extremal surface (of class C'') of (1); it satisfies the Euler equations

$$\frac{\partial}{\partial u}(F_{x_{iu}})+\frac{\partial}{\partial v}(F_{x_{iv}})=0,\qquad i=1, \cdots, n,$$

so that there exists a "conjugate" surface $x_i=\bar{x}_i(u, v)$, determined up to an additive constant (vector), such that

$$\frac{\partial\bar{x}_i}{\partial u}=-\frac{\partial F}{\partial x_{iv}},\quad \frac{\partial\bar{x}_i}{\partial v}=\frac{\partial F}{\partial x_{iu}}. \tag{2}$$

The following facts are demonstrated in this paper: (a) the conjugate surface (2) is independent of the parametric representation of the given extremal of (1); (b) there exists a unique conjugate variational problem

$$\delta\int\int \bar{F}(\bar{p}_{ik})dudv=0 \tag{3}$$

of the same type as (1) with the property that the conjugate surface (2) of an extremal of (1) is an extremal of (3), and conversely; (c) if $S:x_i=x_i(u, v)$ is an extremal of (1), if $\bar{S}:x_i=\bar{x}_i(u, v)$ is its conjugate surface, and if $\bar{\bar{S}}:x_i=\bar{\bar{x}}_i(u, v)$ is the conjugate of $\bar{S}$, then (except for a constant additive vector) $\bar{\bar{x}}_i(u, v)=-x_i(u, v)$, $i=1, \cdots, n$, if all have parametric representations connected by (2), corresponding linear elements on S and $\bar{S}$ being perpendicular in this case. These results are generalizations of similar results proved by A. Haar [Math. Ann. **100**, 481–502 (1928)] for the case $n=3$ and extended to the case $n=4$ by L. Koschmieder [Math. Z. **41**, 43–55 (1936)]. *C. B. Morrey.*

Reid, William T. The Jacobi condition for the double integral problem of the calculus of variations. Duke Math. J. **5**, 856–870 (1939). [MF 825]

Let A be a bounded open connected set whose boundary is C. For simplicity, we say that a function $z(x, y)$ is of class C' on $A+C$ if it is continuous on $A+C$ and if its derivatives z_x and z_y exist at each point of A and coincide with functions continuous on $A+C$. Let $f(x, y, z, p, q)$ be defined and of class C'' on an open region R in (x, y, z, p, q)-space; we say that z is admissible if it is of class C' on $A+C$ and the values

of x, y, z, z_x and z_y are in R for each (x, y) on $A+C$. Suppose $Z(x, y)$ minimizes $\iint_A f(x, y, Z, Z_x, Z_y)dxdy$ among all admissible functions coinciding with it on C and that Z is nonsingular, that is, that $f_{pp}f_{qq}-f_{pq}^2>0$ on $A+C$.

The author extends Jacobi's condition as follows: If $\mathfrak{A}$ is any open connected subset of A whose boundary is $\mathfrak{C}$, there exists no function u of class C' on $\mathfrak{A}+\mathfrak{C}$ with $u\equiv 0$ on $\mathfrak{C}$, $u\not\equiv 0$ on $\mathfrak{A}$ and $|u_x|+|u_y|\not\equiv 0$ on $\mathfrak{C}\cdot A$, and which satisfies the Haar form of the Jacobi equation

$$\text{(A)}\quad \int_\Gamma (f_{pp}u_x+f_{pq}u_y+f_{pz}u)dy-(f_{pq}u_x+f_{qq}u_y+f_{qz}u)dx = \iint_\Delta (f_{pz}u_x+f_{qz}u_y+f_{zz}u)dxdy$$

for every region Δ of $\mathfrak{A}$ whose boundary Γ consists of a finite number of simple closed regular curves in $\mathfrak{A}$. If these equations possess an elementary solution of class C' on $\mathfrak{A}+\mathfrak{C}$ except at the pole, then the requirement that $|u_x|+|u_y|\equiv 0$ on $A\cdot\mathfrak{C}$ may be dropped. Concerning the equations of type (A), the author shows that if (1) the coefficients are continuous on a bounded region $\mathfrak{A}$ and its boundary $\mathfrak{C}$, (2) $f_{pp}f_{qq}-f_{pq}^2>0$ on $\mathfrak{A}+\mathfrak{C}$, (3) an elementary solution exists of class C' on $\mathfrak{A}+\mathfrak{C}$ except at the pole, (4) $m(\mathfrak{A}_k)\leq k\cdot N$, N independent of k, $\mathfrak{A}_k$ denoting those points of $\mathfrak{A}$ whose distance from $\mathfrak{C}$ is not greater than k, then the only solution of (A) of class C' on $\mathfrak{A}+\mathfrak{C}$ and vanishing on $\mathfrak{C}$ is identically zero. (It is not assumed here that f_{pp}, etc., are the derivatives of any function.) Extensions of some results are made to problems involving several dependent and more than two independent variables. *C. B. Morrey.*

Shiffman, Max. The Plateau problem for non-relative minima. Ann. of Math. **40**, 834–854 (1939). [MF 326]

Let Γ be a rectifiable simply closed curve $\mathfrak{x}=\mathfrak{q}(\theta)$ such that the inner product $[\mathfrak{q}(\theta+\Delta\theta)-\mathfrak{q}(\theta)]\cdot[\mathfrak{q}(\varphi+\Delta\varphi)-\mathfrak{q}(\varphi)]$ is non-negative if $\theta-\varphi$, $\Delta\varphi$ and $\Delta\theta$ are near zero and the last two are of the same sign. The first principal theorem is as follows. If Γ bounds two minimal surfaces which give proper relative minima to the Dirichlet integral, it bounds at least one minimal surface which does not give a proper relative minimum to the Dirichlet integral. It is first shown that in the space of piecewise smooth surfaces there is a continuum C_m which contains the two minimizing surfaces and which has the least possible l.u.b. for the Dirichlet integral when compared with all other such continua. This is established by retracting the space of piecewise smooth surfaces into the subspace of potential surfaces. Next it is proved that C_m contains a surface for which the value of the Dirichlet integral equals its l.u.b. on C_m; this requires some care and effort. By the same methods, the Morse inequalities are established for the total type numbers of blocs of minimal surfaces bounded by Γ, a bloc being a maximal connected set of minimal surfaces on which the Dirichlet integral is constant. *E. J. McShane.*

Shiffman, Max. The Plateau problem for minimal surfaces of arbitrary topological structure. Amer. J. Math. **61**, 853–882 (1939). [MF 283]

The paper is concerned with the most general form of the problem of Plateau as first stated and discussed by Douglas. That is, the topological structure as well as the boundary (consisting of a finite number of non-intersecting Jordan curves) of the minimal surface are prescribed. Also, the solution is required to be mapped conformally upon some Riemann surface of the prescribed topological structure. In this paper, Riemann surfaces are taken as slit domains in the $(u+iv)$-plane with properly identified boundary points. To give an account of the main result, let us use the following symbols: K: the class of all slit domains of the prescribed topological type; $[S]$: the class of all surfaces $S:\mathfrak{x}=\mathfrak{x}(u, v)$, where $\mathfrak{x}(u, v)$ is piece-wise of class C' in its range of definition which is some slit domain $\mathfrak{G}$ of class K, and which are bounded by the given set of Jordan curves; $D(\mathfrak{x})$: the integral of $(\mathfrak{x}_u^2+\mathfrak{x}_v^2)/2$ over $\mathfrak{G}$; $[S]_\alpha$: the class of all surfaces of $[S]$ whose inner diameter is equal to α, where the inner diameter is defined as the greatest lower bound of the oscillation of $\mathfrak{x}(u, v)$ on nonbounding closed curves in $\mathfrak{G}$; $[S]_{\alpha\beta}$: the class of all surfaces of $[S]$ whose inner diameter is comprised between α and β, where $0<\alpha<\beta$; $d_\alpha, d_\beta, d_{\alpha\beta}$: the greatest lower bounds of $D(\mathfrak{x})$ in the classes $[S]_\alpha$, $[S]_\beta$, $[S]_{\alpha\beta}$, respectively. The main theorem of the paper states that, if $d_{\alpha\beta}<d_\alpha$, $d_{\alpha\beta}<d_\beta$, then there exists a minimal surface in the class $[S]_{\alpha\beta}$. This minimal surface is obtained as a solution of the problem $D(\mathfrak{x})=\text{minimum}$, stated in the class $[S]_{\alpha\beta}$. The condition $\alpha>0$ is seen to be essential in preventing the solution from degenerating into surfaces of topological structures lower than that prescribed. This same condition $\alpha>0$ makes it possible to avoid the use of the theory of conformal mapping. The case $\alpha=0$, corresponding to the problem of an absolute minimum, would require modifications and is not considered in the paper. The author gives a summary of the relevant literature and states that his methods were mainly inspired by the work of Courant. *T. Radó.*

GEOMETRY

Marchetti, Luigi. Sul problema della trisezione dell'angolo. Period. Mat. **19**, 221–225 (1939). [MF 917]

Sprague, R. Beispiel einer Zerlegung des Quadrats in lauter verschiedene Quadrate. Math. Z. **45**, 607–608 (1939). [MF 413]

Hofmann, Jos. E. Ein neuer Beweis des Morleyschen Satzes. Deutsche Math. **4**, 589–590 (1939). [MF 528]

An indirect and not very short, but clever proof of the proposition: If the trisectors of the internal angles of a triangle are drawn so that those adjacent to each side intersect, the three intersections are the vertices of an equilateral triangle. *N. A. Court* (Norman, Okla.).

van Buuren, C. L. The theorem of Morley. Mathematica, Zutphen. B. **8**, 33–43 (1939). (Dutch) [MF 85]

Four elementary proofs of Morley's theorem on the trisectors of a triangle. *O. Bottema* (Deventer).

Loria, Gino. Triangles équilatéraux dérivés d'un triangle quelconque. Math. Gaz. **23**, 364–372 (1939). [MF 357]

Gulasekharam, F. H. V. Mr. Gibbins' triangle. Math. Gaz. **23**, 360–363 (1939). [MF 356]

Nannei, E. Sulla costruzione dell'ettagono regolare e di altri poligoni regolari di n lati ($n>7$). Period. Mat. **19**, 245–259 (1939). [MF 900]

Delens, Paul. Sur quelques nouvelles acquisitions de la géométrie du tétraèdre. J. Math. Pures Appl. **18**, 303–321 (1939). [MF 809]

The author makes the shrewd, if simple, observation that the angles of a triangle are also the angles which the sides make with the circumcircle, and as such have obvious analogues in the geometry of the tetrahedron; furthermore, these angles are readily expressible in terms of the elements of the tetrahedron. The method of inversion, supported by the methods of Möbius and Grassmann, thus enables the author to attach to a tetrahedron a Brocard angle, a Brocard sphere, a Lemoine point, two Lemoine spheres, two sets of Apollonian spheres, etc. The article under review is a summary of a more detailed study published by the author in Mathesis **51**, 119–127, 444–456 (1937) and **52**, 62–79 (1938). *N. A. Court* (Norman, Okla.).

Gambier, Bertrand. Couples de tétraèdres de Möbius. Ann. École Norm. **56**, 71–118 (1939). [MF 383]

If q_0 is an arbitrary quadric in [3], Δ, Δ' two generators of the same system and T a self-polar tetrahedron, the biaxial involution with axes Δ, Δ' transforms T into another tetrahedron T_1 self-polar with respect to q_0, and such that T and T_1 are in the Möbius position; conversely, every pair T, T_1 of Möbius tetrahedra can always be obtained in this way, and in only one manner. As the 8 vertices A_l and the 8 faces α_l $(l=1, 2, \cdots, 8)$ of any pair of Möbius tetrahedra can be taken in four different ways as the vertices and faces of a similar pair, we obtain altogether four covariant quadrics q_i $(i=0, 1, 2, 3)$, related to the same pair of lines Δ, Δ' lying on them; any two of these quadrics intersect further in another pair of skew lines, axes of another involution which transforms the points A_l and the planes α_l into themselves. It is also possible, and in three different ways, to divide the 8 points A_l into four pairs, so that the four joins of these lie on a single quadric; the three quadrics Σ_j $(j=1, 2, 3)$ obtained in this way being also definable by the dual process. Two quadrics like q_i and Σ_j intersect in 4 lines, are harmonically inscribed in and circumscribed to one another, and their polarities are permutable. There are ∞^2 quadrics φ passing through the 8 points A_l and ∞^2 quadrics φ_1 touching the 8 planes α_l, the two ∞^2 systems described by φ and φ_1 having in common precisely the 3 quadrics Σ_j. In every pencil $\varphi+\lambda\varphi_1$ determined by a quadric of the first and a quadric of the second system there are four cones, corresponding to 4 values of λ, such that the sum of two of them is equal to the sum of the remaining two; conversely, if two quadrics satisfy this condition there are ∞^4 Möbius pairs inscribed in the former and circumscribed to the latter. There are ∞^2 Möbius pairs inscribed in the intersection of two φ and circumscribed to a φ_1; and ∞^1 which are at the same time inscribed in two φ and circumscribed to two φ_1. Any general elliptic twisted quartic admits three different ∞^3 systems of inscribed pairs of Möbius tetrahedra.

The whole research is conducted with elementary methods; it is completed with some unmethodical group considerations, and with the study of some questions connected with the configuration formed by four tetrahedra which are two by two in the Möbius position. *B. Segre.*

Tummers, J. H. Remarkable points and lines of a conic section. Mathematica, Zutphen. B. **8**, 53–57 (1939). (Dutch) [MF 86]

On the Frégier-point of a point of a conic with respect to the conic. Constructions by means of the right-angle instrument. *O. Bottema* (Deventer, Netherlands).

Bouwkamp, C. J. Focal lines of conic sections in connection with a paper by Laguerre. Nieuw Arch. Wiskde **20**, 59–71 (1939). (Dutch) [MF 565]

A conic section is the locus of the points for which the square of the distance to a point has a given ratio to the product of the distances to two given lines. The author, dealing with a special case in Nieuw Arch. Wiskde **19**, 19–30 (1936) has called these lines "focal lines." In the present paper he points out the relations with results obtained by Laguerre [Oeuvres II, Sur la théorie des foyers]. See also Salmon, Conic Sections, 6th ed., pp. 240–241.

D. J. Struik (Cambridge, Mass.).

Bath, F. On circles determined by five lines in a plane. Proc. Cambridge Philos. Soc. **35**, 518–519 (1939). [MF 553]

The known properties of the circles of the figure are proved synthetically and very elegantly by placing the plane in a four dimensional space. *N. A. Court.*

Battaglia, Antonio. Di una generazione geometrica delle cubiche razionali circolari. Period. Mat. **19**, 271–273 (1939). [MF 903]

✶**Locher, Louis. Urphänomene der Geometrie.** Orell Füssli, Zürich, 1939. xvi+164 pp. RM 3.60.

This introduction to projective geometry is intended for those who, with little formal training in mathematics, may wish to become acquainted with the "newer geometry." The treatment is synthetic and largely descriptive, few proofs being offered. The axioms of connection are discussed in some detail, though no attempt is made to give a complete set of postulates. The book is quite readable, presenting in not too hurried fashion the fundamental forms, axioms of connection, infinite elements, decompositions of space (with a discussion of the unilateral character of the projective plane), the theorem and configuration of Desargues (considered as a plane section of a complete space five-point), Moebius nets, and the fundamental theorem. A short chapter on Cassinian ovals closes the book. Many excellently drawn figures stimulate the reader's interest.

The title of the book is explained by the author's feeling that the basic propositions of geometry, far from being arbitrary whims of the geometer, represent an abstract echo of the real world. The word Urphänomene (used by Goethe) seems to the author to be more suggestive of the "true" rôle of these propositions than the word "axiom" with the arbitrariness which this term connotes. Another departure is the appellation "principle of polarity" applied to the duality principle. *L. M. Blumenthal* (Columbia, Mo.).

✶**Prüfer, Heinz. Projektive Geometrie.** Akademische Verlagsgesellschaft m.b.H., Leipzig, 1939. vii+314 pp. RM 9.50.

This book arose from lectures that the author (deceased) gave at Münster University. Synthetic projective geometry is presented based upon an axiomatic structure comprised of postulates of connection (including the parallel postulate, in Playfair's form) for three dimensions, postulates of order and continuity. The operations of projection and section are defined in terms of the primitive notions of the system, and the first five chapters are concerned with developing

in a rigorous manner the projective geometry of forms of the first and second orders in one, two, and three dimensions.

Chapter VI is devoted to metric geometry. The passage from projective to metric geometry is accomplished by the introduction of the concept "line g is orthogonal to plane E," and the addition of postulates involving this notion. Elementary congruence properties are established which form, together with the earlier axioms, a system logically equivalent to Hilbert's foundations of Euclidean geometry. The metric properties of conics are studied without the introduction of imaginary elements. Non-Euclidean geometry is treated in a thoroughly satisfactory manner in Chapter VII, the axioms of orthogonality playing a leading rôle in the development. Chapter VIII contains the foundations of descriptive geometry. The treatment here is probably too concise for any but the exceptional student. In the closing chapter coordinates (in one and two dimensions) are introduced and a brief sketch of analytic projective geometry given.

This is a carefully written book. The figures are plentiful and well drawn, the style and typography are pleasing, and an adequate index is provided. The book contains no exercises. *L. M. Blumenthal* (Columbia, Mo.).

Kubota, Tadahiko. Ein neuer Aufbau der euklidischen Geometrie in der affinen Ebene. Monatsh. Math. Phys. **48**, 96–102 (1939). [MF 629]

Es wird geometrisch bewiesen, dass eine Minkowskische Metrik mit einer Ellipse als Eichkurve euklidisch ist.

H. Busemann (Swarthmore, Pa.).

Gutiérrez Novoa, L. Relations of co-Euclidean geometry. Bol. Mat. **12**, 226–231 (1939). (Spanish) [MF 694]

The author shows that the geometry constructed by him in a previous paper [Bol. Mat. **11**, no. 19] can be regarded as dual to the plane Euclidean geometry, substituting two lines for the two circular points. *E. Helly.*

Steck, Max. Ein Minimalmodell einer endlichen ebenen projektiven Inzidenzgeometrie und die Unabhängigkeit der beiden schwachen Stetigkeitsaxiome von den Inzidenzaxiomen. Math. Z. **45**, 609–634 (1939). [MF 414]

The finite projective plane with 57 points and 57 lines (8 points on each line and 8 lines on each point) satisfies a complete system of incidence axioms for the real projective geometry of the plane. This finite plane is minimal with respect to satisfying all these axioms, for no such plane with fewer points satisfies them all. An example shows that "weak continuity" axioms requiring that, with respect to a nondegenerate conic in a plane, each point of the plane be of one of the following types: (1) a point on just one tangent to the conic, (2) a point on exactly two tangents, (3) a point such that every line on it cuts the conic in two distinct points, are independent of these purely incidence axioms. *J. L. Dorroh* (Arkadelphia, Ark.).

Maier, Karl. Die Desarguessche Konfiguration. Deutsche Math. **4**, 591–641 (1939). [MF 529]

It is well known that the Desargues configuration 10_3 (of ten points lying by threes on ten lines) is a plane section of the lines and planes joining five points of general position in space, and that there is a real or imaginary "reciprocating conic" (Ordnungskurve) with respect to which the ten points are the poles of the ten lines. The author develops these ideas in the following manner. In real projective space, consider a "complete pentagon," consisting of five points $1, \cdots, 5$ of general position joined by ten lines $12, \cdots, 45$ and ten planes $123, \cdots, 345$. Let $[12]$ be the point of intersection of 12 with 345, and let $\langle 12\rangle$ be the harmonic conjugate of this point with respect to 1 and 2. Then the ten points $\langle 12\rangle, \cdots, \langle 45\rangle$ are the vertices of a "complete pentahedron," that is, the points of intersection of five planes $\langle 1\rangle, \cdots, \langle 5\rangle$ (which correspond to the points $1, \cdots, 5$ in a certain elliptic polarity). By taking different plane sections of the complete pentagon, we obtain ∞^3 10_3's. By taking the pencil of planes through any particular line in the plane 123, we select ∞^1 10_3's whose reciprocating conics are sections of a definite quadric (for which $123\langle 45\rangle$ is a self-polar tetrahedron). By taking different lines in the plane 123, we obtain a bundle of such quadrics, passing through the eight associated points $4, 5, [i4], [i5]$ $(i=1, 2, 3)$. Since 123 is one of ten planes $i\,k\,l$, there are ten such bundles of quadrics. Among the various sections of the complete pentagon, those which contain vertices of the dual pentahedron are "symmetrical" in the sense that the 10_3 is transformed into itself by a group of collineations. This group is of order $g=2, 4, 6, 12$, or 24, according as the plane contains one vertex $\langle i\,k\rangle$, contains two vertices $\langle i\,k\rangle$ and $\langle m\,n\rangle$, contains two vertices $\langle i\,k\rangle$ and $\langle k\,l\rangle$, contains three vertices $\langle i\,k\rangle, \langle k\,l\rangle, \langle m\,n\rangle$, or coincides with one of the planes $\langle i\rangle$. (The cyclic permutation (12345) cannot be realized as a collineation.) The general case, when the plane contains no vertex of the pentahedron, may be included by writing $g=1$.

A Desargues configuration is said to be degenerate if two of its points coincide, or if four are collinear. Nondegenerate 10_3's fall into six projectively distinct classes according to the location of the reciprocating conic among the ten points. There may be no real conic; but if the conic is real, either one, two, three or four of the ten points may be interior points of the conic; and if three, these may or may not be collinear. Two 10_3's belonging to the same class may not be projectively related, but they are transformable into one another by continuous variation without becoming degenerate on the way. There are altogether ∞^3 projectively distinct types of 10_3; each type can be realized by $120/g$ plane sections of the complete pentagon. Dually, each type is represented by a set of $120/g$ points in a "configuration space" which is obtained by taking certain cross-ratios as coordinates. The "symmetrical" types are represented by points lying on the ten planes $i\,k\,l$ of a complete pentagon in the configuration space. The degenerate types are represented by points lying on fifteen planes which divide the configuration space into $5+10+60+30+60+20$ cells. (This partition of the 185 cells corresponds to the above classification of 10_3's according to the location of the reciprocating conic.) *H. S. M. Coxeter* (Toronto, Ont.).

Bol, G. Zur Theorie der konvexen Körper. Jber. Deutsch. Math. Verein. **49**, 113–123 (1939). [MF 679]

H. Geppert [Math. Z. **42**, 238 (1936–37)] has extended inequalities, due to Brunn and Minkowski for convex regions, to "supportable" regions ("stützbare Bereiche"), that is, regions defined by a function of support or representable as linear combinations of convex regions. For such regions lengths of arc and areas, if defined by the ordinary expressions, may become negative. The author simplifies Geppert's proofs, and also gives some new applications. The fundamental inequalities $F_{12}^2-F_{11}F_{22}\geqq 0$ and $O_{12}^2-O_{11}O_{22}\geqq 0$ for the mixed areas of two plane regions or of two surfaces in space are derived under the assumption that the linear family determined by them contains elements of

positive area. As an application of the first inequality the isoperimetric inequality of Bonnesen $L^2-4\pi F\geqq\pi^2(r_u-r_i)^2$ is obtained for plane convex regions. A new analogous formula $M^2-4\pi O\geqq(L_M-L_m)^2$ for a convex surface in space is derived from the second inequality, where M is the total mean curvature, O the total area, and L_M and L_m the greatest and least perimeter of a plane projection. *F. John.*

Haupt, Otto. Über Kongruenzregelflächen endlicher Ordnung. Monatsh. Math. Phys. **48**, 245–267 (1939). [MF 643]

Flächenstück $\mathfrak{F}$=lokal kompakte zweidimensionale Punktmenge im dreidimensionalen projektiven Raum R_3, auf der die Punkte, in denen $\mathfrak{F}$ die [mengentheoretische] Dimension zwei hat, dicht liegen. Ein Kongruenzflächenstück baut sich aus den Geraden einer linearen Kongruenz [mit zwei reellen Leitlinien] auf. Ein $\mathfrak{F}$ heisst von endlicher Ordnung, bzw. von der Ordnung n, wenn jede nicht ganz in $\mathfrak{F}$ gelegene Gerade $\mathfrak{F}$ nur in endlich vielen, bzw. höchstens n, Punkten trifft. Ordnung eines Punktes von $\mathfrak{F}$=Minimum der Ordnungen aller geradlinigen Teilflächenstücke von $\mathfrak{F}$, in deren Innern der Punkt liegt. Es werden Struktursätze und Ordnungskriterien für Kongruenzflächenstücke $\mathfrak{F}$ von endlicher Ordnung aufgestellt und insbesondere diejenigen $\mathfrak{F}$ diskutiert, deren sämtliche Punkte dieselbe Ordnung haben. Diese Ordnung ist notwendig höchstens gleich drei. (Ferner Angabe aller geradlinigen Flächenstücke, deren sämtliche Punkte dieselbe Ordnung eins oder zwei haben.) Es wird mit der obigen Definition der Ordnung eines Punktes von $\mathfrak{F}$ eine andere vergleichen, die statt von geradlinigen Teilflächenstücken von beliebigen Umgebungen des Punktes auf $\mathfrak{F}$ ausgeht. Im Zusammenhang damit wird die Existenz und Eindeutigkeit von Schmieghyperboloiden an $\mathfrak{F}$ diskutiert. Indem der Verfasser die Geraden der Kongruenz (zunächst auf die Punkte einer geradlinigen Fläche zweiter Ordnung im R_3 dann diese durch Zentralprojektion) auf die Punkte der Euklidischen Ebene abbildet, führt er seine Sätze auf solche über Bögen in der Ebene zurück. *P. Scherk* (Watertown, Conn.).

Algebraic Geometry

Emch, Arnold. New point configurations and algebraic curves connected with them. Bull. Amer. Math. Soc. **45**, 731–735 (1939). [MF 334]

Soient $P_1(a_1, a_2, a_3)$ et $P_1'(a_2a_3, a_3a_1, a_1a_2)$ deux points homologues quelconques dans la transformation birationnelle plane C représentée par les formules $\rho x_i'=1/x_i$ $(i=1,2,3)$. En échangeant de toutes les manières possibles une des coordonnées de P_1 avec la coordonnée correspondante de P_1' on obtient trois couples de points $P_2(a_2a_3, a_2, a_3)$, $P_2'(a_1, a_3a_1, a_1a_2)$; $P_3(a_1, a_3a_1, a_3)$, $P_3'(a_2a_3, a_2, a_1a_2)$; $P_4(a_1, a_2, a_1a_2)$, $P_4'(a_2a_3, a_3a_1, a_3)$ qui se correspondent dans la même transformation. Les 8 points P_iP_i', les points fondamentaux des coordonnées, les points $B(1,1,1)$ et $B_1(-1,1,1)$, $B_2(1,-1,1)$, $B_3(1,1,-1)$ et le point $O(a_1+a_2a_3, a_2+a_3a_1, a_3+a_1a_2)$ appartiennent à une même courbe C du troisième degré, et forment sur C une configuration remarquable. Par exemple, les 4 droites P_iP_i' passent par O et les 8 points sont deux à deux en ligne droite avec chacun des points fondamentaux des coordonnées. La courbe C est invariante dans la transformation C. Dans l'espace on obtient d'une manière analogue, à partir de la transformation $\rho x_i'=1/x_i$ $(i=1, 2, 3, 4)$, une configuration Δ_{16} de 16 points $P_iP_i'(i=1, \cdots, 8)$. Les 8 droites P_iP_i' passent par le point $O(a_1+a_2a_3a_4, a_2+a_3a_4a_1, a_3+a_4a_1a_2, a_4+a_1a_2a_3)$. Les droites joignant deux points homologues quelconques dans la transformation considérée forment un complexe du troisième degré; le cône du complexe ayant pour sommet un point quelconque de l'espace contient toujours une courbe invariante d'ordre 7 et genre 3, qui contient à son tour une simple infinité de configurations telles que Δ_{16}; dans toutes ces configurations le sommet du cône joue le rôle du point O. *E. G. Togliatti* (Genova).

Cherubino, Salvatore. Identità birazionale di due curve algebriche. Rend. Sem. Mat. Roma **3**, 1–22 (1939). [MF 507]

In the preface it is stated that this paper was originally intended for the use of the author's students in the University of Pisa. It is a compendium with numerous references to the work of several mathematicians. It deals chiefly with the conditions for the birational identity of two algebraic curves. In the last two sections, curves with the same jacobian variety are treated. The work is often more detailed and in some places more precise than in the original papers. *T. R. Hollcroft* (Aurora, N. Y.).

Apéry, Roger. Sur les sextiques à 8 rebroussements. C. R. Acad. Sci. Paris **209**, 744–746 (1939). [MF 841]

By means of the residue theorem and the related results of the theory of the algebraic curves, the following theorem is proved: The necessary and sufficient condition that there exists a plane algebraic curve of the sixth order with cusps in eight given points $A_1, A_2, \cdots, A_8$, is that these eight points can be divided in four couples of points so that the two points of every couple correspond to each other in the same harmonic homology, and that the four lines through the corresponding points of these couples form an equianharmonic set. *E. Helly* (New York, N. Y.).

Amin, Amin Yasin. Sur une méthode géométrique permettant d'obtenir 991 des 2015 coniques de contact d'une courbe plane quintique. C. R. Acad. Sci. Paris **209**, 337–338 (1939). [MF 234]

On sait que si une surface Q du 4e ordre est touchée par un plan π le long d'une conique γ, elle possède sur γ six points doubles; soit O l'un d'eux; le cône circonscrit à Q du point O se décompose dans le plan π et dans un cône C du 5e ordre; le cône C et le cône du 2e ordre tangent à Q au point O ont en commun cinq génératrices de contact. On peut construire encore 990 cônes du 2e ordre qui touchent C le long de cinq génératrices de la manière suivante. Il y a ∞^4 surfaces du second ordre passant par γ; elles découpent sur Q, à part γ, des courbes du 6e ordre; chacune de ces courbes est la ligne de contact de Q avec une surface du 3e ordre convenable; d'après un théorème connu, dû à G. Fano, il y a 495 parmi ces surfaces du 3e ordre qui possèdent quatre points doubles; les cônes du 4e ordre circonscrits à ces surfaces particulières du point O se décomposent chacun en deux cônes du second ordre; on obtient ainsi les $2\times495=990$ cônes cherchés. *E. G. Togliatti.*

Bompiani, Enrico. Intorno ad alcune superficie razionali del 4° ordine. Boll. Un. Mat. Ital. **1**, 305–314 (1939). [MF 571]

Starting from two recent notes on the rational quartic surfaces $F_4^{(2)}$, $F_4^{(3)}$ of Noether, G. Conforto [Atti Accad. Naz. Lincei. Rend. (6) **29**, 20–24, 43–48 (1939)] shows that from differential properties an $F_4^{(3)}$ is a subcase of $F_4^{(2)}$. From the curve of branch points of the representative double planes, S. Faedo [Boll. Un. Mat. Ital. (2) **1**, no. 3

(1939)] obtains that an $F_4^{(3)}$ is a particular case of $F_4^{(2)}$. The present paper, by more elementary methods, shows that both these notes deal with particular cases, neither general surface is a particular case of the other. They have as subcase a series of quartics with a double line in common.
V. Snyder (Ithaca, N. Y.).

Bompiani, E. Über zwei Kalotten einer Hyperquadrik. Jber. Deutsch. Math. Verein. **49**, 143–145 (1939). [MF 683]

If a line r meets a plane algebraic curve of order n in n points P_i at which the curvature is ρ_i and the angle which the tangent makes with r is τ_i, then the relation

$$\sum_{i=1}^{n}\frac{1}{\rho_i\sin^3\tau_i}=0.$$

is satisfied [Blaschke and Boll, Geometrie der Gewebe, 1938, 221–222]. No satisfactory generalization of this theorem to higher dimensions has been made. The present paper shows that if r meets a quadric primal in $[k]$ in P_1, P_2 then

$$\frac{K_1}{\cos^{k+1}(n_1,\tau)}=(-1)^{k-1}\frac{K_2}{\cos^{k+1}(n_2,\tau)},$$

K_i being the total curvature at P_i, and n_i the normal.
V. Snyder (Ithaca, N. Y.).

Babbage, D. W. A note on the quadrics through a canonical curve. J. London Math. Soc. **14**, 310–315 (1939). [MF 438]

The canonical curve C of order $2(p-1)$ in S_{p-1} of a non-hyperelliptic plane curve C' of genus $p\geqq 4$ is the complete intersection of a system Q of $\frac{1}{2}(p-2)(p-3)$ linearly independent quadric primals except in two cases: (1) If C' has a g_3^1, the corresponding series on C is cut out by a system of trisecant lines generating a rational, normal ruled surface of order $p-2$ common to Q. (2) If $p=6$ and C' has a g_5^2, the corresponding series on C is cut out by a system of quinquesecant planes generating a Veronese quartic surface common to Q. In this paper, the proof of the above theorem, only sketched heretofore [F. Enriques and O. Chisini, Teoria geometrica delle equazioni e delle funzioni algebriche, vol. 3, 97–108 (1924)], is given in detail, including the elimination of an additional possibility that Q might contain loci not through C. *T. R. Hollcroft.*

Todd, J. A. A remark on a theorem of Severi. Proc. Cambridge Philos. Soc. **35**, 516–517 (1939). [MF 552]

Using a recent result obtained by Hodge [J. London Math. Soc. **12**, 58 (1937)], the author proves the following theorem due to Severi [Math. Ann. **62**, 194 (1906)]: If P and Q are two virtual curves on an algebraic surface such that

$$(P^2)=(PQ)=(Q^2)=n>0,$$

then there exists an integer λ such that $\lambda P=\lambda Q$; without Severi's restriction that either P or Q must have no components of negative virtual grade. *T. R. Hollcroft.*

Pompilj, Giuseppe. Sulla rappresentazione algebrica dei piani tripli. Rend. Sem. Mat. Roma **3**, 109–132 (1939). [MF 505]

The general triple plane is shown to be the projection from the point at infinity on the z-axis of a surface of the form $z^3+3pz+q=0$, where p, q are polynomials of order $2m$, $3m$ in x, y. If $4p^3+q^2$ has a squared factor not dividing p, q, this is no part of the branch curve, but the projection of the double curve of the surface. If there is a common factor r-fold in p and s-fold in q, we can suppose without loss of generality that either $r<2$ or $s<3$; in this case the corresponding curve is a simple or double constituent of the branch curve (locus of two sheet or three sheet branching) according as $3r<2s$ or $3r>2s$, and is also the projection of a double curve of the surface if $s>1$. It is shown that there exist polynomials a, b, c, d, g, of orders k, $h+m$, $2(h+m)-k$, $3(h+m)-2k$, h, such that

$$pg^2=ac-b^2,\quad qg^3=a^2d-3abc+2b^3,\quad 4p^3+q^2=a^2\phi,$$

where $\phi=0$ is the branch curve. Conditions for $h=0$ are discussed, and it is shown that in this case the branch curves for given m and k ($\leqq 3m/2$) are a single continuous system with $6m^2-3km$ cusps. (This expression for the number of cusps is clearly independent of h, though the author does not seem to say so.)

A point which is r-ple for $p=0$ and s-ple for $q=0$ is ρ-ple for the branch curve, where $\rho\geqq\min(3r, 2s)$, equality holding unless $3r=2s$. It is not a branch point if $\rho\equiv 0 \pmod 6$, and is a two sheet branch point if $\rho\equiv 1, 3, 5 \pmod 6$, and a three sheet branch point if $\rho\equiv 2, 4 \pmod 6$.
P. Du Val (Manchester).

Franchetta, Alfredo. Su alcuni esempi di superficie canoniche. Rend. Sem. Mat. Roma **3**, 23–28 (1939). [MF 508]

The author extends to surfaces in [3] the method given by Enriques [Math. Ann. **85**, 195 (1922)] for linear systems of plane curves by first constructing a rational surface having an autoresidual characteristic series of its plane representation and a certain point singularity, the fundamental image of which in the plane representation is supposed non-existent. Examples of surfaces, normal in [3], for which $p_g=4$, $p^{(1)}=6, 7, \cdots, 11$ are constructed. *V. Snyder.*

Macpherson, R. E. Canonical systems on a reducible variety. Proc. Cambridge Philos. Soc. **35**, 389–393 (1939). [MF 544]

A recent paper by J. Todd [Proc. London Math. Soc. (2) **43**, 127–138 (1937)] establishes a property of adjunction for the general canonical varieties of any dimension $k<d$, on a nonsingular V_d, cut out by nonsingular varieties S_{d-1}, forming a linear system. By means of a generalization of the formula for the genus of a composite curve, applied to dimensions 3 and 4 by L. Roth [Proc. Cambridge Philos. Soc. **29**, 88–94 (1933)], the author extends the same formula to varieties S not in a linear system. *V. Snyder.*

Morin, Ugo. Sulla classificazione proiettiva delle varietà a superficie-sezioni razionali. Ann. Mat. Pura Appl. **18**, 147–171 (1939). [MF 587]

The projective classification of irreducible algebraic varieties M_r, of order n and of dimension $r\geqq 3$ in S_m, with rational, elliptic and hyperelliptic curve sections was made by various authors near the beginning of this century. The present paper gives a similar classification of the M_r with curve sections of any given genus and with rational surface sections. Seven projectively distinct types of such M_r are obtained for $r\geqq 3$ with additional types for certain particular values of r. All of the M_r so classified are rational except the cubic primal. *T. R. Hollcroft* (Aurora, N. Y.).

Todd, J. A. The geometrical invariants of algebraic loci. (Second paper.) Proc. London Math. Soc. **45**, 410–424 (1939). [MF 386]

In a former paper by the same title [Proc. London Math. Soc. (2) **43**, 127–141 (1937)], the author defines on an algebraic variety V_d of dimension d invariant systems $\{X_h\}$ of varieties of dimension h, $0\leqq h\leqq d-1$, which he calls canonical systems, and bases the proof of their existence on an assumed criterion of equivalence for varieties on V_d. After the above paper was written, Eger [C. R. Acad. Sci. Paris **204**, 92–94, 217–219 (1937)] defined similar canonical systems from a function-theoretic point of view, but gave no existence proofs. In the present paper, which is intended to replace the paper cited first above, the existence of $\{X_h\}$ is established by geometric methods without making use of any unproved assumption. *T. R. Hollcroft.*

Differential Geometry

De Cicco, John. The differential geometry of series of lineal elements. Trans. Amer. Math. Soc. **46**, 348–361 (1939). [MF 468]

The group of this geometry is the whirl-motion group G_6 in the plane [see E. Kasner, Amer. J. Math. **33**, 193–202 (1911), and Kasner-De Cicco, Bull. Amer. Math. Soc. **44**, 399–403 (1938)]. A series is a set of ∞^1 lineal elements. This paper gives an account of the differential geometry of such series by considering its tangent turbines and osculating flat fields. This leads to the notion of curvature k and torsion t of a series and the intrinsic equations of a series of lineal elements under the G_6 in the classical form $k=k(s)$, $t=0$, where s is the arc-length of the point-union between the initial element and any element.

D. J. Struik (Cambridge, Mass.).

De Cicco, John. The analogue of the Moebius group of circular transformations in the Kasner plane. Bull. Amer. Math. Soc. **45**, 936–943 (1939). [MF 786]

With the third order differential elements tangent at a point is associated the Kasner x, y plane, where x is the curvature and $y=dx/ds$. If "distance" is Kasner's conformal invariant, the "circles" are given by $a_0x^2+a_1x+a_2y+a_3=0$. The author studies point transformations of the Kasner plane and their effect on these "circles." Among other results, it is shown that the transformations which take all "circles" into "circles" is the continuous seven parameter group given by

$$X=\frac{ax+b}{cx+d},\quad Y=\frac{ey+fx^2+gx+h}{(cx+d)^2}.$$

P. Franklin (Cambridge, Mass.).

Varela Gil, J. Determination of the total curvature of a closed curve. Bol. Mat. **12**, 201–205 (1939). (Spanish) [MF 457]

It is shown that the total curvature of a closed curve equals $\pi\cdot\sum_{i=1}^{r}\epsilon_i$, where r designates the number of tangents parallel to an arbitrary but fixed straight line and $\epsilon_i=0$, $+1$, -1 according to whether the tangent crosses the curve at the point of contact or stays on the one or the other side of the curve. *H. Busemann* (Swarthmore, Pa.).

Löbell, Frank. Eine räumliche Verallgemeinerung des Vierscheitelsatzes. Jber. Deutsch. Math. Verein. **49**, 140–143 (1939). [MF 682]

Der Autor beweist unter Benutzung eines Satzes von ihm über n-Tupel von richtungstreu aufeinander bezogenen Ovalen [Jber. Deutsch. Math. Verein. **48**, 172 ff. (1938)] den folgenden Satz: Sei τ eine geschlossene Raumkurve mit stetiger (nicht konstanter) Krümmung; ihr sphärisches Tangentenbild t entspreche ihr eineindeutig. Fällt der Schwerpunkt S von t in den Mittelpunkt O der t tragenden Einheitskugel, so gebe es durch jedes Punktepaar von t eine Ebene, die t in zwei durch sie getrennte Teile zerlegt. Fällt S nicht mit O zusammen, so sei die Projektion von t parallel zu OS ein Oval. Dann hat die Krümmung von τ mindestens zwei Maxima, die grösser, und mindestens zwei Minima, die kleiner sind als der Mittelwert der Krümmung längs τ. [Der Satz kann leicht direkt bewiesen, und, da die Definition der Krümmung nur bei der Definition von S benutzt wird, auf beliebige stetige Funktionen auf τ übertragen werden.] *P. Scherk* (Watertown, Conn.).

Rosca, Radu. Transformations asymptotiques des courbes de l'espace elliptique. Courbes de Bertrand. J. Math. Pures Appl. **18**, 167–215 (1939). [MF 382]

On appelle transformées asymptotiques deux courbes de l'espace elliptique $x^h(t)$, $\bar{x}^h(t)$ $(h=0, 1, 2, 3)$ se correspondant ponctuellement, de sorte que la droite joignant les points homologues soit l'intersection des plans osculateurs correspondants (definition de Bianchi). En même que la courbe (x), l'auteur considère la courbe décrite par le point y, situé sur la binormale en x à la distance $\pi/2$. Si les courbes (x), $(\bar{x})$ se correspondent asymptotiquement, les deux courbes (y), $(\bar{y})$ sont aussi en correspondance asymptotique. Les rayons de courbure et torsion et l'arc de $(\bar{x})$ sont obtenu en fonction des éléments de la courbe (x). On obtient deux cas, où les distances $(x\,\bar{x})$ et $(y\,\bar{y})$ sont constantes toutes deux: celui des courbes de Bertrand et celui des courbes à torsion constante. Les propriétés importantes de ces courbes et des courbes minima dans l'espace elliptique sont développées. *J. Haantjes* (Amsterdam).

Su, Buchin. Plane sections of the tangent surface of a space curve. Ann. Mat. Pura Appl. **18**, 77–96 (1939). [MF 584]

The author investigates plane sections C of the tangent surface T of a projective space curve Γ, in particular the case in which the tangent to Γ at the point P does not lie in the sectioning plane. Basic in the discussion are a certain projectivity $\mathfrak{B}$ and polarity $\mathfrak{P}$ determined by a neighborhood of the fourth order of Γ at P. The various "Popa osculants" of C are studied [L. Popa: Geometria proiettivo-differenziale delle singolarità delle curve piane, Atti Accad. Naz. Lincei. Rend. **25**, 220–222 (1937)]. Typical theorem: Consider all sections $\{C\}$ of T by planes through a given line t_n (through P, in the osculating plane of Γ at P and not tangent to Γ at P). Then every section has a cusp at P and the covariant line of Popa determined by each C describes a plane ω_n. The correspondence established between t_n and ω_n is the projectivity $\mathfrak{P}\cdot\mathfrak{B}$. The last part of the paper applies the earlier results to investigating certain configurations projectively related to a pair of intersecting space curves, in particular the case in which the two curves are asymptotic curves of a surface. *J. L. Vanderslice.*

Bell, P. O. The first canonical pencil. Duke Math. J. **5**, 784–788 (1939). [MF 818]

Let there be given a line l lying in the tangent plane π of a surface S at a point P. The polar reciprocal l' of l with respect to the quadric of Lie intersects the quadric of

Wilczynski and the quadric of Lie in two points τ and ω. As P moves along a curve C on S the points τ and ω describe curves whose tangents intersect π in points of a line r. The line r intersects the tangent t to C in a point ν. The point ν depends only on the direction of t at P. The locus of ν as t varies through the pencil with center P is a line $\bar{l}$. The locus of the point μ of intersection of l and $\bar{l}$ for all lines l in π is the reciprocal of the projective normal. Let t' be the tangent line at P conjugate to t, l' and $\bar{l}'$ the reciprocals of l and $\bar{l}$, respectively. The lines t', l', $\bar{l}'$ are coplanar and the harmonic conjugate of t' with respect to l' and $\bar{l}'$ is the projective normal. The last section of the paper gives a geometrical characterization of a general line l_k of the canonical pencil of the first kind. There is a pencil of quadrics D_{hk} having second order contact with the surface S at P which intersects S in a curve whose triple point tangents are tangent to the curves which lie on both of its osculating asymptotic quadrics and a quadric D_k of Darboux. The polar line of the projective normal with respect to D_{hk} is a canonical line l_k of the first kind depending on k only.
V. G. Grove (East Lansing, Mich.).

Bell, P. O. A study of curved surfaces by means of certain associated ruled surfaces. Trans. Amer. Math. Soc. **46**, 389–409 (1939). [MF 471]

The tangents to the curves of first one and then the other of the families of asymptotic curves at the points of a given non-asymptotic curve on a surface through a point P form two ruled surfaces. These surfaces may be so paired with the asymptotic ruled surfaces that the surfaces of each pair have at P an asymptotic tangent as a common generator. The tangent plane to one ruled surface of the pair at a point on the common generator is tangent to the other surface of the pair at a unique point on the generator. There is thus associated with a point on each asymptotic tangent a unique point. This latter point depends only on the direction of the given non-asymptotic curve. In this manner to a given line (or congruence of lines) in the tangent plane there is made to correspond a line (or congruence of lines) in the tangent plane, the latter depending uniquely on a parameter λ. This correspondence, and correspondences obtained from it by polar reciprocation with respect to the quadric of Lie, and the quadrics of Darboux enables the author, among other things, to give new geometrical characterizations of the curves of Darboux and Segre, of the Darboux-Segre pencils of conjugate nets, of the curves corresponding to the developables of lines lying in the tangent plane, of the union curves of a congruence of lines, to generalize the transformation of Čech, to further characterize the correspondence of Segre, the pangeodesics, the projective normal, the hypergeodesics, and the union curves associated therewith, and finally the directrices of Wilczynski and the edges of Green. *V. G. Grove.*

Simpson, Harold. An extension of Savary's theorem. J. London Math. Soc. **14**, 315–318 (1939). [MF 439]

A surface S rolls upon an applicable surface S' so that the single point of contact P traces out curves C and C' whose points correspond in the applicability. A point O rigidly fixed in S will describe a roulette R of S with respect to S'. By means of the moving trihedral it is shown that OP is normal to R at O and expressions for the principal centers of curvature of the roulette are obtained.
M. S. Knebelman (Pullman, Wash.).

Calugareanu, Georges. Sur les surfaces de M. Tzitzéica qui sont des surfaces de révolution. Bull. Sect. Sci. Acad. Roum. **20** (1938), 173–175 (1939). [MF 616]

A surface of Tzitzéica is defined to be such that the total curvature at any point M is proportional to the fourth power of the distance from the origin to the tangent plane at M. Such surfaces have application to the theory of deformation of surfaces. If the surface is also a surface of revolution, the differential equation of the meridian is $y'y''=\lambda x(y-xy')^4$. The author effects, by a sequence of substitutions based upon the group of invariance of the differential equation, its integration in terms of elliptic integrals, thus effecting a simplification of the method of integration of Tzitzéica, which is based on more general principles. *J. W. Green* (Rochester, N. Y.).

Ansermet, A. L'emploi en géodésie de coordonnées polaires conformes. Schweiz. Z. Vermessgswes. **37**, 235–240 (1939). [MF 591]

Tonolo, A. Trasporto delle coordinate geografiche e dell'azimut lungo un arco di linea qualunque di un ellissoide di rotazione. Atti Accad. Naz. Lincei. Rend. **29**, 573–580 (1939). [MF 276]

In this paper the author develops the latitude, longitude and azimuth of a point on an ellipsoid of revolution as power series in the arc of an arbitrary curve on the surface. The coefficients appearing in these expansions are functions of the latitude, longitude, azimuth, the radii of curvature of the parallel of latitude and the meridian of longitude and the geodesic curvature of the arbitrary curve through the given point on the ellipsoid. *V. G. Grove.*

van Kampen, E. R. A remark on asymptotic curves. Amer. J. Math. **61**, 992–994 (1939). [MF 296]

Given a section of a surface S of negative Gaussian curvature in ordinary three dimensional space, and P a point of S in this section. Let that part of the intersection of S and its tangent plane at P which is in a small angular vicinity of the asymptotic direction l be a curve γ of class C^2 and let γ have a non-vanishing curvature in a vicinity of P (not at P). Then the asymptotic curve tangent to l is separated by the curve from its tangent at P. *D. J. Struik.*

Grünbaum, Siegfried. Über die Bestimmung von Flächen aus ihrer Normalkrümmung längs einer Schar geodätischer Linien. Comment. Math. Helv. **12**, 71–74 (1939). [MF 500]

If $k(u, v)$ is an analytic function of the two variables, there exists a real analytic surface for which k is the normal curvature of a family of geodesics orthogonal to the curve $u=0$, u being the arc length on the geodesic $v=$const. The surface is uniquely determined by $k(u, v)$ and the curve $u=0$. If $u=0$ is a point, we have the case of geodesic polar coordinates and the surface is again analytic if k satisfies Euler's equation. The proof is based on the fact that the Cauchy-Kowalevsky conditions of integrability of the equations of Gauss and of Codazzi-Mainardi are satisfied.
M. S. Knebelman (Pullman, Wash.).

Jaeger, C. G. A class of surfaces applicable to the sphere. Amer. Math. Monthly **46**, 410–416 (1939). [MF 202]

In this paper the equations of a unit sphere are taken to be $x=\cos u\cdot\cos v$, $y=\cos u\cdot\sin v$, $z=\sin u$. Surfaces applicable to this sphere are considered for which the coeffi-

cients of the second fundamental form are functions of u alone. But even in this special case the paper gives nothing new; the guessed solution of equations (9) to (14) is based on known results and it is not verified that this solution satisfies equations (9), (13) and (14).

M. S. Knebelman (Pullman, Wash.).

Miranda, Carlo. Su un problema di Minkowski. Rend. Sem. Mat. Roma **3**, 96–108 (1939). [MF 504]

The author treats the problem of determining a convex surface S of given Gaussian curvature K as a function of the oriented normal. Minkowski showed the existence in a certain average sense. The reviewer proved the existence of an analytic S if K is analytic [Trans. Amer. Math. Soc. **43**, 258–270 (1938)]. In this paper the author supposes K to have derivatives of mth order satisfying Hölder conditions of exponent $\lambda>0$, and concludes the existence of an S which has derivatives of order $m+2$. His tools are certain theorems about compactness of solutions of elliptic equations and an a priori estimate of the mean curvature of S which yields bounds for the second derivatives of S.

H. Lewy (Berkeley, Calif.).

Buzano, Piero. Determinazione e studio di superficie di S_5 le cui linee principali presentano una notevole particolarità. Ann. Mat. Pura Appl. **18**, 51–76 (1939). [MF 583]

Plane webs are considered in the Geometrie der Gewebe by Blaschke and Boll, 1938. The 5-webs that have triads in hexagonal position may be mapped on systems of asymptotic lines on a surface F of five-space [5]. Terracini [Atti Accad. Naz. Lincei. Rend. (6) **25** (1937)] considered the case in which three of the systems on F are conical curves, or cut from it by systems of [3]'s through a fixed plane, with the plane vertices of the cones all in one [4].

The author considers a system with the last restriction removed. He first discusses a system of four partial differential equations of the third order which defines F and considers conditions of integrability and makes various transformations which preserve topological properties. All the solutions are derived. The solutions are all reducible to two projectively distinct types of surfaces in [5], A and B. These systems have in common: (1) 5 systems of asymptotic lines I, II, $\cdots$, V in a fixed [4]. (2) I, II are separated harmonically by IV, V. (3) Systems I, II, III are cut by [3]'s each tangent to a cone having a plane for vertex; these planes are incident two by two, but do not lie in a fixed [4]. (4) Systems IV and V lie on a quadric cone of [3], those of IV having a fixed vertex, those of V another fixed vertex. (5) The image webs of the triads in I, II, IV, V; I, II, III; III, IV, V are all hexagonal.

V. Snyder.

Vasseur, Marcel. Déformation d'une surface avec un réseau conjugué permanent dans l'espace elliptique. C. R. Acad. Sci. Paris **209**, 823–825 (1939). [MF 859]

Let S be a surface in elliptic space, and let the parametric lines on S form a conjugate net. The author seeks the surfaces S' obtained from S by isometric deformation, with the requirement that the parametric curves retain their property of forming a conjugate net. The coefficients e' and g' of the second fundamental form of S' ($f'=0$) satisfy the same equations of Gauss-Codazzi as do e and g of S. The set of solutions of these equations yields all surfaces S'. The author studies the equations and obtains results analogous to those obtained in Euclidean space. He remarks that there can be obtained in this way three different types of surfaces S', due to different types of solutions of the Gauss-Codazzi equations.

J. W. Green (Rochester, N. Y.).

✶Takasu, Tsurusaburo. Differentialgeometrien in den Kugelräumen. Bd. II. Laguerresche Differentialkugelgeometrie. Maruzen Company, Ltd., Tokyo, 1939. xx+444 pp.

This book, planned as 3 volumes, consists mainly of a unified systematic exposition of the author's results in the differential-geometries in sphere-spaces. Volume II contains mostly original contributions to Laguerre's differential sphere-geometry, only about a fifth of the volume being devoted to the results of other authors. The method adopted is based on the following fundamental principle, which was conjectured in 1920 and established in 1925 by the author: The Laguerre-space is an Euclidean space which is provided with a variable dual space-curvature as well as with moving absolute circles, so that angles are measured with variable units. Approximately a third of the material is taken from papers of the author which have appeared in Japanese journals. About half of the book may be considered as original reports, so that there is little overlapping with Blaschke's "Vorlesungen über Differentialgeometrie," Bd. III (1928). The contents are as follows:

1. Einleitende Theorie. §1. Einleitung. §2. Minimalprojektion. Ein Gegenstück der T-Poincaré-Klein-Repräsentierung. §3. Euklidische Auffassung der Laguerreschen Geometrie.

2. Theorie der L-Kreisscharen. §4. Allgemeine Theorie der L-Kreisscharen. §5. Allgemeine Theorie der Kurven in der L-Ebene. §6. Verschiedene Sätze aus der Theorie der L-Kreisscharen. §7. L-Kreisscharen und L-Kurven. §8. L-Kreisscharen und Kurven im Grossen.

3. Theorie der L-Kugelscharen. §9. Allgemeine Theorie der L-Kugelscharen. (L-Verallgemeinerung der allgemeinen Kurventheorie im Raume.) §10. Allgemeine Torsentheorie im Laguerreschen Raume. (L-Verallgemeinerung der einzelnen Theorien der speziellen euklidischen Torsen.) §11. Verschiedene Sätze aus der Theorie der L-Kugelscharen. §12. Spezielle L-Kugelscharen und spezielle L-Torsen. §13. L-Kugelscharen im Grossen.

4. Theorie der L-Kugelkongruenzen. §14. Laguerre-geometrische Verallgemeinerung der Grundlage der Flächentheorie. §15. L-Konjugiertes System. L-Hauptstreifen. L-Hauptorthogonalstreifen. §16. Krümmungsstreifen, L-Hauptkrümmungsstreifen, Blaschkesche Mittenkugeln. Eine natürliche Normierungsweise der L-Flächenebene. §17. L-Flächenkrümmungen. §18. L-geodätische Krümmungen und L-geodätische Torsionen. §19. Zyklische Torsensysteme auf einer Fläche und ihre Analoga. §20. Merkwürdige L-Flächentorsen und merkwürdige L-Kugelkongruenz-L-Kugelscharen. §21. Einige Sätze über Kugelkongruenzen. §22. Grundlage der L-Flächentheorie und die der Theorie der Mittenkugelkongruenzen. (L-Verallgemeinerung der Theorie der Minimalflächen.) §23. Grundlage der L-Flächentheorie und der Theorie der L-Krümmungskugelkongruenzen. (L-Verallgemeinerung der Theorie der abwickelbaren Flächen.) §24. L-Kugelkongruenzentheorie und L-Flächentheorie in Bonnetschen Ebenenkoordinaten. §25. Laguerresche Flächentheorie mit den L-Tangentialkugeln als Elementen, die eine Ribaucoursche L-Kugelkongruenz bilden. §26. Eine kürzere Begründung der Laguerreschen Flächentheorie. §27. L-Torsen Blaschkes. §28. Einige Sätze über die ebenen und sphärischen

Krümmungslinien. §29. Über die Nabelpunkte. §30. L-Minimalflächen. §31. L-Minimalflächen vom Gesichtspunkte der Bewegungsgeometrie aus. §32. $(\Delta+f)^n$-Mittenkugelkongruenzen und $(\Delta+f)^n$-L-Minimalflächen. §33. Minimal-L-Kugelkongruenzen.

5. Theorie der L-Kegelsysteme. §34. Theorie der allgemeinen L-Kegelkongruenzen und Theorie der L-Facettenkongruenzen. §35. Laguerre-geometrische Verallgemeinerung der Konstruktion der L-Minimalflächen von de Montcheuil.

Most of the fundamental results of Euclidean differential geometry are generalized Laguerre-geometrically. In particular, the theory of L-minimal surfaces, which had been investigated by W. Blaschke, is developed further and almost all results of the theory of the ordinary minimal surfaces are Laguerre-geometrically generalized.

A. Kawaguchi (Sapporo).

Kowalewski, Gerhard. Ein Beitrag zur projektiven Differentialgeometrie. Monatsh. Math. Phys. **48**, 1–18 (1939). [MF 620]

This paper is concerned with the projective differential geometry of plane curves. The method is that of natural geometry (Cèsaro, Pick, Kowalewski) and makes strong use of relative coordinates. Let $\mathfrak{e}$ represent a general curve element of the sixth order (not belonging to a conic) and J the projective curvature. Path curves, $J=\text{const.}$, are introduced and singular path curves defined. The singular curve through any $\mathfrak{e}$ is unique and projectively equivalent to $y=x\log x$. The two ends of the curve are called the first and second pole of $\mathfrak{e}$. The locus of the first (second) pole as $\mathfrak{e}$ traverses a curve C is called the first (second) lower evolute of C. The curves C with linear first (second) evolute are determined. With each $\mathfrak{e}$ is associated an osculating conic and two special cubics. One of these cubics is used to define higher evolutes, both, to define the projective normal. Harmonic properties of the tangent, normal, osculating conic and poles of $\mathfrak{e}$ are derived. The equation of the projective normal at $\mathfrak{e}$ is given explicitly in terms of the eight coordinates of $\mathfrak{e}$. The paper ends with the determination (quotient of power series) of the curves whose projective normals are concurrent.

J. L. Vanderslice.

Fon, Te-Chih. Note on the projective differential geometry of space curves. Ann. Mat. Pura Appl. **18**, 97–106 (1939). [MF 585]

A study is made of the fundamental tetrahedron used by Sannia in his derivation of the projective analogue of the Frenet-Serret formulae [G. Sannia: Nuova trattazione della geometria proiettivo-differenziale delle curve sghembe, Ann. Mat. Pura Appl. **1**, 1–18 (1924) and **3**, 1–25 (1926)]. The geometrical construction of this tetrahedron is simplified by the use of a certain class of covariant quadrics associated with the given space curve. Furthermore, important projective invariants such as the first and second projective curvatures are expressed completely in terms of cross-ratios.

J. L. Vanderslice (Bethlehem, Pa.).

Rasmusen, Ruth B. The canonical lines and the extremals of two invariant integrals. Amer. J. Math. **61**, 1004–1008 (1939). [MF 298]

The terminology and basic ideas are to be found in E. P. Lane, Projective Differential Geometry, chap. 3. Two new geometric properties of the canonical lines at a point on a surface are given, also a geometric characterization of the hypergeodesics defined on a surface by certain [Lane, p. 117, prob. 27] invariant integrals.

J. L. Vanderslice.

Popa, Ilie. Géométrie projective différentielle du point conique des surfaces. Bull. Sect. Sci. Acad. Roum. **20** (1938), 79–83 (1939). [MF 614]

In this paper the author considers the projective geometry of a surface in the neighborhood of a conical point 0, in the special case in which the cone is a nondegenerate quadric. A canonical form of the power series expansion for the surface is derived. Geometrical characterizations are given for the tetrahedron of reference and unit point giving rise to the canonical expansion. Attention is called to the interdependence of the invariant coefficients in the expansion. The cone at the point 0 intersects the surface in a curve; the six tangents to this curve are called the principal tangents. A correspondence is set up between a principal tangent and a unique point, and between the tangent plane to the cone along a principal tangent and a certain line. Other correspondences are set up between the lines of the bundle at the point 0.

V. G. Grove.

Michal, Aristotle D. et Mewborn, Aladuke Boyd. Géométrie différentielle projective générale des géodésiques généralisées. C. R. Acad. Sci. Paris **209**, 392–394 (1939). [MF 376]

A formal generalization of the theory of Berwald [Ann. of Math. **37**, 870–898 (1936)] of the projective geometry of paths into a Hausdorff topological space H, for which the allowable coordinates $K^{(3)}$ are in a Banach space. A projective connection $\Pi(X, Y, Z)$ is defined by the relation

$$\Pi(X, Y, Z)=[\Gamma(x, y, z)+Mz^0y+My^0z,\ \Gamma^0(x, y, z)+My^0z^0]$$

in a Banach space of pairs $X=(x, x^0)$ from a class of linear symmetric connections $\Gamma(x, \xi_1, \xi_2)$ and a class of the scalar gauge forms $\Gamma^0(x, \xi_1, \xi_2)$, which submit to any projective change of connection and to any change of gauge form, respectively, where M is a positive number and

$$x^0=-\frac{1}{2M}\log\frac{ds}{d\pi}$$

is a gauge variable. The projective normal parameter π is determined by the Schwarzian derivative

$$\{\pi, s\}=-2M\,\Gamma^0\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right)$$

from an affine parameter s. Three fundamental theorems are also stated without proof, and no reference is made to the theory of curvature.

A. Kawaguchi (Sapporo).

Kawaguchi, Akitsugu. Eine Verallgemeinerung von Extensoren. Monatsh. Math. Phys. **48**, 329–339 (1939). [MF 650]

Consider the equations $x^i=x^i(u^1, \cdots, u^\kappa)$, where $i=1, \cdots, N$, which define (locally) a κ-dimensional surface in an N-dimensional manifold and also the derivatives of the functions $x^i(u)$, namely, $p^i_{\alpha_1}, p^i_{\alpha_1\alpha_2}, \cdots, p^i_{\alpha_1\ldots\alpha_M}$, where the α's are the indices of differentiation. On account of the transitive property of the equations of transformation of the p's, induced by transformations of the coordinates x, one can evidently construct a theory of invariants, closely related to but generalizing the theory of tensors, in which the p's play the rôle of independent variables. The details of this formally complicated theory for an arbitrary but

fixed value of the above integer M have been worked out by Kawaguchi. Many of the ordinary terms associated with the theory of tensors have been used and some new ones added appropriate to the complications here involved. The work generalizes the theory of extensors by H. V. Craig which corresponds to the case $\kappa=1$. Complete references are given to related papers. *T. Y. Thomas.*

van der Kulk, W. Uebertragungen mit alternierendem Krümmungsaffinor. Nederl. Akad. Wetensch., Proc. **42**, 753–763 (1939). [MF 422]

Necessary and sufficient conditions are given that, in a L_n (notation of Schouten-Struik),

$$R_{\nu(\mu\lambda)}^{\cdots\kappa}=\frac{1}{n+1}(V_{\nu(\mu}-R_{\nu(\mu})A_{\lambda)}^{\kappa},$$

and in particular that

$$R_{\nu(\mu\lambda)}^{\cdots\kappa}=0,$$

in which case the curvature tensor is alternating in ν, μ, λ. The case that L_n is semisymmetrical $(\Gamma_{[\mu\lambda]}^{\kappa}=S_{[\mu}A_{\lambda]}^{\kappa})$ receives special attention. The results are expressed with the aid of, and also without, cinematical conceptions.

D. J. Struik (Cambridge, Mass.).

Coburn, N. V_m in S_n with planar points ($m\geqq 3$). Bull. Amer. Math. Soc. **45**, 774–783 (1939). [MF 343]

Given a Riemannian manifold of m dimensions V_m in an n-dimensional space of constant curvature S_n of which all points are planar. In this case the curvature properties of the V_m depend on two second fundamental tensors. If the rank of any of these two tensors is greater than 2, then these V_m are (1) V_m consisting of ∞^1 V_{m-1} imbedded in ∞^1 S_{m+1}, or (2) V_m consisting of ∞^1 V_{m-1} imbedded in ∞^1 S_m, or (3) V_m lying in S_{m+2}. This theorem can be considered as a generalization of a theorem by C. Segre on V_m in S_n with only axial points. *D. J. Struik.*

Thomas, T. Y. Reducible Riemann spaces and their characterization. Monatsh. Math. Phys. **48**, 288–292 (1939). [MF 646]

An interesting illustration of the new methods used in the geometry of Riemann spaces in the large is given in this paper. In an analytic and topologically connected Riemann space R of n-dimensions, the following terms are defined: (1) a metric tensor $T_{\alpha\beta}$ is any symmetric tensor whose covariant derivative vanishes; (2) the index t of R is the maximum number of linearly independent metric tensors; (3) R is said to be reducible if it is isometric to the product of two or more Riemann spaces. First, the ∞^1 integrability conditions for the vanishing of the covariant derivative of any metric tensor (denoted by E_0, E_1, E_2, etc.) are considered. Since the number of unknowns $T_{\alpha\beta}$ contained in the system E is $k=n(n+1)/2$, the infinite matrix of this system has rank $r<k$ at any point P. By use of normal coordinates and some of the general methods developed by the author, it is shown that r is constant over R. The author and W. Mayer [Compositio Math. **5** (1937)] have shown that the rank of the infinite system E coincides with that of the finite system $E_0\cdots E_k\rightarrow M'$. Hence the index of R is $t=k-r$, where r is the rank of the finite system M'. Now let H_m denote the set of equations obtained by equating to zero all determinants in M' of order $k-m+1$ at point P. By Gram's theorem, H_m can be expressed by the vanishing of one or more tensor differential invariants of R. If the squares of each of these invariants is formed and added (in a locally Euclidean orthogonal coordinate system at P), then a single scalar invariant is obtained $G_m=0$. Thus if and only if $r\leqq k-m$, that is, $t\geqq m$, is $G_m=0$. It has been shown by the author [Monatsh. Math. Phys. **47** (1939)] that R is reducible if and only if $t\geqq 2$. Hence if and only if $G_2=0$ is R reducible. *N. Coburn* (Austin, Tex.).

Ficken, F. A. The Riemannian and affine differential geometry of product-spaces. Ann. of Math. **40**, 892–913 (1939). [MF 329]

The paper deals almost entirely with the ordinary product, called here the direct product, of two Riemann spaces (positive definite quadratic form). Products of tensors, geodesic subspaces, parallel displacement, parallel fields of vector spaces and motions are considered. A general product for which the metric is not uniquely determined by the metrics of the factor spaces is discussed as an extension of the direct product. The product of two affinely connected spaces is defined and mention is made of results in the theory of the direct product of Riemann spaces which can be carried over to the product of affinely connected spaces.

Among the principal results having to do with the direct product of Riemann spaces are the following: A space of constant non-vanishing curvature cannot be a product space. $R=R_p\times R_q$ is conformally flat if, and only if, (a) when $p=1$, $q\geqq 1$, R_q is of constant curvature and (b) when $p\geqq 2$, $q\geqq 2$, R_p and R_q have constant curvatures K_1 and K_2 such that $K_1+K_2=0$. Here p and q denote the dimensionality of R_p and R_q, respectively. The holonomic group of $R=R_p\times R_q$ is the direct product of the holonomic groups of R_p and R_q. Two decompositions of R into simple factor spaces differ at most in the order of the factors. By a simple factor space is meant here one which cannot be decomposed further into factors. The results of this paper are entirely of local character. [References: Duschek-Mayer, Lehrbuch der Differentialgeometrie II, pp. 147–152; L. P. Eisenhart, Trans. Amer. Math. Soc. **25**, 297–306 (1923); H. Levy, Ann. of Math. **27**, 91–98 (1926). For a general discussion of Riemann product spaces in the large see T. Y. Thomas, Monatsh. Math. Phys. **47**, 388–418 (1939).]

T. Y. Thomas (Los Angeles, Calif.).

Thomas, T. Y. Imbedding theorems in differential geometry. Bull. Amer. Math. Soc. **45**, 841–850 (1939). [MF 769]

In this expository review the author outlines a new, comparatively simple proof that an n-dimensional manifold of class C^r ($r\geqq 2$) may be mapped topologically and in a C^r manner in a $(2n+1)$-dimensional Euclidean space, a theorem due to H. Whitney, who proved it for $r\geqq 1$. Also mentioned are the outstanding results concerning the isometric imbedding (occasionally referred to in this paper as isomorphic imbedding) of Riemannian spaces locally in Euclidean spaces, particularly Thomas' own algebraic characterization of n-dimensional simply connected Riemannian manifolds which are isometrically mappable as hypersurfaces in an $(n+1)$-dimensional Euclidean space.

C. B. Tompkins (Princeton, N. J.).

Yano, Kentaro. Sur la théorie des espaces à connexion conforme. J. Fac. Sci. Imp. Univ. Tokyo. Sect. 1. **4**, 1–59 (1939). [MF 605]

Following Cartan [Les espaces à connexion conforme, Ann. Soc. Polon. Math. **1923**], the author begins by associating an ordinary conformal n-space referred to poly-

spherical coordinates with each point of a general underlying n-space. A conformal connection is introduced to establish an infinitesimal conformal mapping between neighboring "tangent spaces." Both Pfaffian and tensor notation are used as convenient. The local coordinate systems are progressively specialized by imposing invariant conditions on them and the connection until there is just one in each tangent space associated with a given system in the underlying space. Simultaneously the connection becomes completely normalized and a conformal tensor calculus arises. This first part of the paper can be characterized as an extensive elaboration of Cartan's work in the light of later developments. Particularly strong use is made of methods and notation suggested by Veblen in two notes in the Proc. Nat. Acad. Sci. U. S. A. [**1928, 1935**]. The rest of the paper is devoted to an exposition of previously published work of the author [Proc. Imp. Acad. Tokyo **1938**]. Any curve has defined upon it a projective parameter invariant up to homographies. With its aid Frenet equations are set up for the generalized conformal space. Generalized circles are defined as curves of vanishing first conformal curvature and their differential equations are obtained in parametric form. For the conformal euclidean case the solution of these equations leads to ordinary circles and an interpretation of the projective parameter. There is a complete bibliography.

J. L. Vanderslice (Bethlehem, Pa.).

Levine, Jack. Groups of motions in conformally flat spaces. II. Bull. Amer. Math. Soc. **45**, 766–773 (1939). [MF 342]

This paper tabulates the 20 types of subgroups of the conformal group in n variables that can serve as a group of motions of a conformally flat space into itself and gives the metric of each of the admissible spaces. It also restates more fully Theorem 2 of paper I with same title [Bull. Amer. Math. Soc. **42**, 418–422 (1936)]. *M. S. Knebelman.*

Haantjes, J. und Wrona, W. Ueber konformeuklidische und Einsteinsche Räume gerader Dimension. Nederl. Akad. Wetensch., Proc. **42**, 626–636 (1939). [MF 317]

The scalar curvature of an m-direction at a point in a Riemannian space V_n is defined as the scalar curvature $R/m(m-1)$ of a V_m geodesic at the given point and tangent to the given m-direction. The following theorems are proved: A necessary and sufficient condition that a V_{2m} be an Einstein space ($R_{ij}=(R/n)g_{ij}$) is that the scalar curvatures of any two mutually perpendicular m-directions be always equal. This is well known for $m=2$. A necessary and sufficient condition that a V_{2m} be conformally flat is (1) that the sum of the scalar curvatures of any two mutually perpendicular m-directions be at every point independent of the choice of these m-directions (the sum is $R/m(n-1)$); or (2) that the sum of the Riemannian curvatures of any system of m mutually perpendicular 2-directions be at every point independent of the system chosen (the sum is $R/(2n-2)$).

For $n=4$ it is shown that two distinct types of totally isotropic 2-directions exist and, corresponding to these, two distinct special types of bivectors (designated respectively as first and second kind). Totally isotropic 2-spaces are distinguished by the type of their tangential bivectors. Several properties of these special bivectors are developed, culminating in the following theorem related to spinor analysis: In a V_4 the differential equation $\phi^{ij}f^{kl}_{;j}=0$ for a bivector field of the first kind, where ϕ^{ij} is the tangential bivector of any totally isotropic 2-space of the second kind, is completely integrable if and only if the space is an Einstein space. *J. L. Vanderslice* (Bethlehem, Pa.).

Berwald, L. Über Finslersche und Cartansche Geometrie II. Invarianten bei der Variation vielfacher Integrale und Parallelhyperflächen in Cartanschen Räumen. Compositio Math. **7**, 141–176 (1939). [MF 374]

This paper is mainly concerned with the properties of the family of parallel hypersurfaces in the n-dimensional Cartan space. By the n-dimensional Cartan space we mean an n-dimensional manifold whose geometry is ruled by a $(n-1)$-ple integral defining the hypersurface volume of a piece of a hypersurface [see E. Cartan: Les espaces métriques . . . , Actual. Sci. Ind. **72** (1933) and L. Berwald, Acta Math. **71**, 191–248 (1939)]. After a short explanation of the fundamental ideas and the important formulae in the Cartan space, the author derives the normal form of the second variation of the integral of hypersurface volume, which was found first by L. Koschmieder [Nederl. Akad. Wetensch., Proc. **31**, 140–150, 469–484 (1927)]. Koschmieder's scalar U_0^* appearing in the normal form is interpreted for a minimal hypersurface in the Riemannian space by means of the curvature tensors $U_0^*=R_{ih}l^il^h-(n-1)(n-2)k$, and the transversal coordinates which play an important rôle in the following theory of parallel hypersurfaces are introduced. Next the ideas of "parallel hypersurfaces" and "breadth-distance" are defined, and the author shows that the well-known properties of geodesically parallel hypersurfaces in the Riemannian geometry hold also for the parallel hypersurfaces in Cartan spaces with a space volume, but not in the general Cartan space. The geometrical meaning of the mean extremal curvature H and the scalar U_0^* is also explained and it is proved that in a regular Cartan space a family of parallel hypersurfaces consists only of hyperplanes when the parallel hypersurfaces are at the same time extremals. *A. Kawaguchi.*

MECHANICS

Mechanics of Continua

Sedov, L. I. Application of the theory of functions of a complex variable to some problems of the plane hydrodynamics. Uspekhi Matem. Nauk **6**, 120–182 (1939). (Russian) [MF 397]

A survey of effective methods of solution of plane problems of hydrodynamics concerning a potential flow of an incompressible liquid. The paper contains numerous bibliographical references.

Seth, B. R. On the motion of a liquid set up by a moving regular polygonal cylinder. J. London Math. Soc. **14**, 255–261 (1939). [MF 430]

The calculation of the stream lines of an infinitely long cylinder with cross-section Q, which is moving in the ideal liquid, can be reduced to the conformal mapping of the exterior of Q into the exterior of a circle. The author discusses the case in which Q is a regular polygon, and therefore the mapping function can be represented by the (generalized) integral formula of Schwarz. A constant which

appears in this formula can be expressed by Γ functions. Using this fact the author gives formulas for the stream lines, the kinetic energy, etc., which are convenient for numerical calculations. The case of a rotating cylinder is treated in an analogous manner. *S. Bergmann.*

Peretti, G. Vortici nei veli liquidi viscosi. Atti Accad. Naz. Lincei. Rend. **29**, 581–584 (1939). [MF 277]

The steady motion of a viscous film moving on a surface in geodesic circles about a point is completely determined. The forces exerted by the film on the surface are shown to have a zero resultant. More special results are obtained for surfaces of revolution. *D. C. Lewis* (Durham, N. H.).

Hylleraas, Egil A. Über die Schwingungen eines stabil geschichteten, durch Meridiane begrenzten Meeres. Astrophys. Norvegica **3**, 139–164 (1939). [MF 613]

The hydrodynamical equations for an ideal incompressible fluid on a rotating sphere, subjected to the influence of tidal forces, are investigated. It is shown that, by assuming a radial density variation of the fluid, the two-dimensional theory of tidal waves can be derived from the general equations. This is said to explain a result by Solberg [Astrophys. Norvegica **1**, 237 (1936)] who was unable to obtain these two-dimensional equations for a homogeneous fluid. The final result of the paper is the formulation of the two-dimensional boundary value problem and of the corresponding problem of the calculus of variations for the pressure distribution in the fluid. The solution of this problem is promised in a subsequent paper. *E. Reissner.*

Tolman, Richard C. On the stability of spheres of simple mechanical fluid held together by Newtonian gravitation. Astrophys. J. **90**, 541–567 (1939). [MF 735]

The author considers a spherical fluid body at rest whose particles are held together by Newtonian gravitation, the pressure-density relation being such that pressure increases with depth at just the rate necessary to support gravitational forces. He uses well-known tests of the calculus of variations to determine the stability of a figure of equilibrium, the condition being that the total potential energy is a minimum. The Euler test, the Legendre test and the Weierstrass test are satisfied. The Jacobi test leads to the discovery of possible instability. He studies in some detail possibilities for the Emden "equation of state" $p=\kappa\rho^\gamma$. He shows that instability occurs when $\gamma<4/3$, a result previously obtained by Landau [Phys. Z. Sowjetunion **1**, 285 (1932)]. He carries through necessary calculations to establish stability for cases $\gamma=\infty$ and $\gamma=2$; and to establish neutral equilibrium for the case $\gamma=4/3$. He observes that for $\gamma<6/5$ the body has infinite density at its center and extends to infinity, but for $\gamma=6/5$ the density may be finite at the center but the body extends to infinity.

E. J. Moulton (Evanston, Ill.).

Tolman, Richard C. On the stability of stellar models, with remarks on the origin of novae. Astrophys. J. **90**, 568–600 (1939). [MF 736]

This paper is closely related to the preceding, but undertakes to include temperature relations in the interior of stars in investigating their secular stability.

E. J. Moulton (Evanston, Ill.).

Morris, Rosa M. The two-dimensional hydrodynamical theory of moving aerofoils. III. Proc. Roy. Soc. London. Ser. A. **172**, 213–230 (1939). [MF 214]

It is first shown that the Joukowski condition reduces to a single equation so that a surmise in II is incorrect. A case of the older theory in which expressions for the force and couple can be obtained in finite form and their dependence on thickness and camber completely specified is then examined in detail. This is the case of the Joukowsky aerofoil defined by the transformation

$$z=ae^{-i\zeta}+b^2/(z_0+ae^{-i\zeta}), \qquad |z_0|<a,$$

in which the origin is taken at the center of the circle from which the aerofoil is transformed. When the transformation is written in the form

$$z=e^{-i\zeta}\sum_{n=0}^{\infty} a_n e^{ni\zeta},$$

the quantities

$$b_n=\sum_{r=0}^{\infty} a_{n+r}\bar{a}_r$$

are easily found. The circulation is determined and expressions are found for the forces and couple on the aerofoil.

The longitudinal stability of the simple rectilinear motion of the aerofoil is then discussed. For simplicity the discussion is restricted to the cases of a flat plate and a nearly flat Joukowsky aerofoil. In the first case the condition for stability is expressed in a simple form. In the second case the condition is a little more complex. It shows that an increase in camber means a reduction of stability in all cases. With a certain assumption regarding the effect of increasing the thickness of the fore-end on the position of the center of gravity, such an increase means an increase of the stability. These results agree with those of previous writers but are a little more precise. *H. Bateman.*

Green, A. E. The forces acting on a circular arc aerofoil in a stream bounded by a plane wall. Proc. London Math. Soc. **46**, 19–54 (1939). [MF 733]

The author investigates the two-dimensional steady irrotational continuous flow of an incompressible inviscid fluid past a circular arc aerofoil which is placed in any position near an infinite plane wall. Taking the plane of the fluid motion as the $z=(x+iy)$-plane, the complex velocity potential $\Omega(z)$ is expressed in the form

$$\frac{d\Omega}{ds}=M\{\wp(s)-\wp(s_1)\}, \qquad \frac{z-\lambda}{z+\lambda}=\frac{\sigma(s_3)\sigma(s-s_4)}{\sigma(s_4)\sigma(s-s_3)}\exp\left(\frac{\eta_1\mu}{\omega_1}s\right),$$

where $\wp$ and σ are the customary elliptic functions and all other quantities, except s, constants depending on the shape and position of the aerofoil. If the x- and y-components of the aerofoil are denoted by X and Y and the moment of the forces about the trailing edge by Γ, then it is known that

$$Y+iX=-\tfrac{1}{2}\rho\int_C\left(\frac{d\Omega}{dz}\right)^2 dz, \quad \Gamma=\tfrac{1}{2}\rho\Re\left\{\int_C\left(\frac{d\Omega}{dz}\right)^2(z-z_A)\,dz\right\},$$

where ρ is the density of the fluid, z_A the coordinate of the trailing edge and C any closed contour surrounding the aerofoil. The evaluation of these integrals leads to complicated expressions in terms of the four theta functions, which are given explicitly in the paper. Some numerical calculations have also been made. The special case when the trailing edge of the aerofoil touches the ground was previously treated by S. Tomotika and I. Imai [Proc. Phys.-Math. Soc. Japan **20**, 15–32 (1938)]. *E. Reissner.*

Wieghardt, Karl. Über die Auftriebsverteilung des einfachen Rechteckflügels über die Tiefe. Z. Angew. Math. Mech. 19, 257–270 (1939). [MF 488]

The paper is a contribution to three-dimensional airfoil theory. In that theory an airfoil is represented as a two-dimensional vortex sheet in an ideal fluid, the shape of the sheet being that of the airfoil. The spatial distribution of vortex strength has to be such that the velocity induced by the vortices at a point on the sheet cancels the component of the flight velocity normal to the plane of the airfoil at that point. The mathematical formulation of this problem leads to a singular integral equation of the first kind for the unknown vortex distribution. The author gives approximate solutions of this integral equation for the case of a rectangular airfoil. The method consists essentially in substituting for the integral equation a system of linear equations which are solved numerically. It is important for convergence purposes to observe in the transition from integral to linear equation certain fluid mechanical principles. The necessary numerical calculations are very extensive and doubts are expressed by the author concerning the accuracy of the results. *E. Reissner.*

Love, A. E. H. Boussinesq's problem for a rigid cone. Quart. J. Math., Oxford Ser. 10, 161–175 (1939). [MF 597]

The problem reduces to that of finding a function V harmonic everywhere except on the pressed area ($z=0$, $\rho<c$) and such that when $z=0$, $V=A(w_0-\rho\cot\alpha)$ for $0\leqq\rho\leqq c$, $\partial V/\partial z=0$, $\rho>c$. The problem is solved with the aid of spheroidal coordinates of the type in which $z+i\rho=c \operatorname{sh}(\eta+i\theta)$. It is found that the condition $w_0=\frac{1}{2}\pi c\cot\alpha$ must be satisfied, otherwise the pressure will be infinite when $\rho=c$. When this condition is satisfied the pressure is given by the expression $2\pi p=AQ_0(\cos\theta)\cot\alpha$. A finite expression for V is found by using the quantities $R_1^2=\rho^2+(z-ic)^2$, $R_2^2=\rho^2+(z+ic)^2$. Expressions are given for derivatives of V which are needed for the calculation of the stresses and displacements. The reviewer remarks that V can also be expressed in the forms

$$\tan\alpha\cdot V=\int_0^\infty e^{-zt}J_0(\rho t)[1-\cos(ct)]dt/t^2$$

$$=\int_0^{\pi/2} d\alpha F[(z^2\tan^2\alpha+\rho^2\sin^2\alpha)^{\frac{1}{2}}],$$

where $F(u)=c-u$, when $0<u<c$, $F(u)=0$, when $u>c$.

H. Bateman (Pasadena, Calif.).

Lennertz, J. Zur Berechnung der Eigenwerte für achsensymmetrische Schwingungen von Hohlzylindern. Z. Angew. Math. Mech. 19, 286–289 (1939). [MF 490]

The differential equation of elastic vibratory motion is

$$G\left[\nabla^2 U-\frac{1}{1-2\nu}\operatorname{grad}\operatorname{div} U\right]=\rho\ddot{U},$$

where U is the displacement vector, G and ν the elastic moduli. The author determines the characteristic values for torsional, radial and longitudinal oscillations of a hollow cylinder. For the coupled axially symmetric radial and longitudinal oscillations, two series of eigenwerte are obtained; the first series compares with the values for purely longitudinal oscillations of a short bar of length equal to that of the cylinder. The second series is obtained for an infinite length cylinder and corresponds to purely radial oscillations. The eigenwerte for the torsional oscillations are obtained from a transcendental equation involving the usual Bessel functions. *D. L. Holl* (Ames, Iowa).

Biot, M. A. Increase of torsional stiffness of a prismatical bar due to axial tension. J. Appl. Phys. 10, 860–864 (1939). [MF 612]

The author's theory of elasticity of second order is applied to calculate the increase of torsional stiffness of a prismatical bar under the action of a constant initial axial tension. Only those second order terms are retained which are of kinematic origin. In this problem it is assumed that stresses are linear functions of the strains and that the local rotation is large compared to the local strain. It is found that the classical shear stress distribution of Saint Venant is also a solution of this problem. The boundary condition differs from the classical torsion problem in that it contains a term involving the product of the axial tension S and the rotations ω_z or ω_y. To the degree of the approximation assumed this furnishes an increase in the torsional stiffness which is proportional to the polar inertia of the section with respect to its center of gravity. This correction is only of significance in thin twisted strips or in thin-walled cylinders split along a generator. In the former the correction is $\frac{1}{4}(b/c)^2(S/G)$, where G is the shear modulus and (b, c) are the dimensions of the rectangular section.

D. L. Holl (Ames, Iowa).

Morrow, John. The bending of thick plates under certain specified conditions of support. Philos. Mag. 28, 73–80 (1939). [MF 731]

The author considers the bending of a thick rectangular plate, the top and bottom surfaces of which are given by $z=\pm h$. Constant normal pressure is applied to the top surface $z=h$. The plate is "clamped" at the edges, that is, the tangent plane of the middle surface ($z=0$) is considered horizontal at all points of the edge of that surface. However, instead of prescribing the deflection of the middle surface at its edge, the requirement is made that the resultant vertical supporting force should be constant along the edge, a rather artificial condition from the physical point of view. Nevertheless, in view of the complexity of the thick plate problem, it is remarkable that the author obtains an exact solution of his problem in the form of simple polynomials in the coordinates x, y, z.

J. J. Stoker (New York, N. Y.).

Friedrichs, K. O. and Stoker, J. J. The non-linear boundary value problem of the buckled plate. Proc. Nat. Acad. Sci. U. S. A. 25, 535–540 (1939). [MF 269]

An exact solution is obtained of the problem of the behavior of a thin circular plate after buckling under a uniform edge thrust in the plane of the plate. The results are valid for an unlimited range of the ratio α of the prescribed edge thrust to that at which buckling begins. A method of perturbations, used in connection with the suitably modified non-linear differential equation of v. Kármán for plates with large deflections, leads to satisfactory results for moderate values of the ratio α. For larger values of this ratio recourse is had to the method employed by S. Way [Trans. Amer. Soc. Mech. Eng. 56, 627–636 (1934)] in studying the boundary of circular plates with large deflections. A noteworthy result is that the central region of the plate is found to be in tension for values of the ratio α greater than 1.57. An asymptotic solution is obtained for large values of the ratio α. The possibility of second and higher buckling states is pointed out. *H. W. March.*

Kappus, Robert. Zur Elastizitätstheorie endlicher Verschiebungen. Z. Angew. Math. Mech. 19, 271–285 (1939). [MF 489]

The purpose of the paper is to develop a general theory of elasticity which will include the classical theory as a limiting case. It represents a systematic development of two papers by Trefftz [Verh. 3. Int. Congr. Tech. Mech. III, 44–50 (1931); Z. Angew. Math. Mech. 13, 160–165 (1933)] which were concerned with questions of elastic stability. The state of deformation in the neighborhood of a point, whose coordinates before the deformation were x, y and z, is described with the aid of three vectors $\mathbf{g}_x$, $\mathbf{g}_y$, $\mathbf{g}_z$. A vector of length dx parallel to the X-axis before deformation becomes the vector $\mathbf{g}_x dx$ after deformation, the components of $\mathbf{g}_x$ being $1+\partial u/\partial x$, $\partial v/\partial x$, $\partial w/\partial x$, where u, v and w denote the components of the displacement of the point in the body originally in the position (x, y, z). In terms of these vectors the nine quantities $g_{xx}, g_{xy}, g_{yx}, \cdots$ are defined as follows: $g_{xx}=\mathbf{g}_x^2-1$, $g_{xy}=\mathbf{g}_x\cdot\mathbf{g}_y, \cdots$. In particular $g_{xy}=g_{yx}$. From their transformation properties it follows that the quantities $g_{xx}, g_{xy}, \cdots$ are the components of a tensor. The actual components of strain, which do not form a tensor unless they are infinitesimal, can be expressed in terms of the quantities g_{xx}, g_{xy}, etc. Associated with three principal directions (before the deformation) are the quantities g_{11}, g_{22}, g_{33}, which attain extreme values, and the quantities g_{12}, g_{23}, g_{31}, which vanish. The stress distribution is described in terms of the forces acting on the faces of the deformed parallelepiped which in the undeformed state was a rectangular parallelepiped with edges dx, dy, dz. These forces are represented by the vectors $\pm\mathbf{k}_x dydz$, $\pm\mathbf{k}_y dzdx$, $\pm\mathbf{k}_z dxdy$. The vectors $\mathbf{k}_x$, $\mathbf{k}_y$ and $\mathbf{k}_z$ are each resolved into components parallel to the vectors $\mathbf{g}_x$, $\mathbf{g}_y$, $\mathbf{g}_z$. The components of $\mathbf{k}_x$, for example, are denoted by k_{xx}, k_{xy}, k_{xz}. It is readily shown that relations such as $k_{xy}=k_{yx}$ hold and that the quantities $k_{xx}, \cdots, k_{yz}$ form a tensor, also that there exist three orthogonally intersecting planes (before deformation) for which $k_{12}=k_{23}=k_{32}=0$ and k_{11}, k_{22}, k_{33} attain extreme values. The actual components of stress $\sigma_x, \cdots, \tau_{xy}$, which do not in general form a tensor, can be expressed in terms of the quantities k_{xx}, k_{xy}, etc. It is no longer true in general that $\tau_{xy}=\tau_{yx}$. From the fact that the quantities g_{11}, g_{22}, g_{33}, k_{11}, k_{22}, k_{33} are invariants, it follows that the most general relation between stress and strain in a homogeneous isotropic body can be expressed by the following equations:

$$g_{11}=f(k_{11};k_{22},k_{33};\Theta),\quad g_{22}=f(k_{22};k_{33},k_{11};\Theta),$$
$$g_{33}=f(k_{33};k_{11},k_{22};\Theta),$$

where Θ denotes the difference in temperature of the initial and deformed states. This general law is replaced by the usual linear relation since the latter holds for most metallic materials so long as the deformations are purely elastic. The equations of equilibrium and the boundary conditions are deduced from a consideration of the forces acting on a deformed element. Further topics treated are the principle of virtual displacements and the elastic potential and its relation to the internal energy. A second part of the paper will contain applications of the theory to the torsion and flexure of rods and in particular to problems in elastic stability. The references to the literature do not include the papers of B. R. Seth [Philos. Trans. Roy. Soc. London, Ser. A. 234, 231 (1935)] and F. D. Murnaghan [Amer. J. Math. 59, 235 (1937)]. *H. W. March* (Madison, Wis.).

Poncin, Henri. Sur les conditions de stabilité d'une discontinuité dans un milieu continu. Acta Math. 71, 1–62 (1939). [MF 210]

Un milieu continu incompressible est limité extérieurement par une surface solide animé d'un certain mouvement; à un instant donné il existe à l'intérieur une surface de discontinuité relative à la densité. Cette surface partage le milieu en deux régions dans chacune desquelles la densité est constante, le mouvement irrotationnel et à plan directeur. L'auteur se propose de déterminer les conditions dans lesquelles cette surface de discontinuité conserve sa forme pendant le mouvement. Si l'on considère le mouvement comme résultant de la composition d'un mouvement d'ensemble et d'un mouvement de déformation, la stabilité de la surface de discontinuité ne peut être assurée que par les mouvements d'ensemble dans lesquelles la rotation instantanée a ou une valeur constante ou une valeur inversement proportionnelle au temps. La surface de discontinuité peut avoir une forme quelconque et le mouvement de déformation dépend des éléments géométriques de cette surface.
W. Prager (Istanbul).

Rock, Donald Hill. Finite strain analysis in elastic theory. Iowa State Coll. J. Sci. 14, 71–72 (1939). [MF 744]

Abstract of a thesis.

Capocaccia, Antonio Agostino. Un metodo per l'analisi dimensionale di sistemi di più grandezze. Atti Soc. Sci. Genova 4, 143–153 (1939). [MF 272]

In the case of systems of nonhomogeneous linear equations, the general solution is the sum of a particular solution and the general solution of the corresponding homogeneous system. This well-known principle, in so far as it pertains to finite systems with more unknowns than equations, is applied to the technique of dimensional analysis.
D. C. Lewis (Durham, N. H.).

Capocaccia, Antonio Agostino. Il principio di similitudine meccanica applicato ai films lubrificanti. Atti Soc. Sci. Genova 4, 154–165 (1939). [MF 273]

An application of the paper reviewed above.
D. C. Lewis (Durham, N. H.).

MATHEMATICAL PHYSICS

Manarini, Mario. Sulle forze ponderomotrici nei dielettrici eterogenei in relazione alle tensioni elastiche nei corpi deformati. Boll. Un. Mat. Ital. 1, 345–350 (1939). [MF 578]

A vectorial derivation of a formula of Maxwell and Helmholtz for electrostatic forces in a nonhomogeneous dielectric is followed by an interpretation in terms of tensions and pressures in directions along the lines of force and perpendicular thereto. *D. C. Lewis, Jr.* (Durham, N. H.).

Agostinelli, Cataldo. Sul moto di un corpuscolo elettrizzato in un campo magnetico simmetrico rispetto a un asse e integrazione del problema in un caso particolare. Atti Accad. Sci. Torino 74, 69–85 (1939). [MF 486]

This paper is a continuation of two earlier papers by the same author on the motion of an electrified particle in the field of a magnetic dipole [Atti Accad. Sci. Torino 73, 460–474 (1938)] and in the field of a number of axially aligned n-poles [Atti Accad. Naz. Lincei. Rend. 28, 88–92 (1938)].

In the present paper the author assumes that the magnetic field is defined by a potential and that it possesses axial symmetry. The differential equations of the problem then reduce to those of a particle moving in a plane subjected to a force defined by a known potential function. If the field is also symmetrical with respect to a plane perpendicular to the axis of symmetry, and if the particle is originally moving in this plane, the integration of the equations can be reduced to a single quadrature. For the particular case of a magnetic dipole the integration is effected in terms of elliptic functions. *M. C. Gray* (New York, N. Y.).

Agostinelli, Cataldo. Sul moto di un corpuscolo elettrizzato in presenza di un dipolo magnetico e in prossimità del piano equatoriale. Ist. Lombardo, Rend. **72**, 285–300 (1939). [MF 926]

This paper is an extension of the above reviewed paper in which the motion of an electrified particle in the equatorial plane of a magnetic dipole was discussed. In the present paper the Poincaré "equations of variation" are used to extend the solution to regions near the equatorial plane. A solution applicable in a narrow strip bounded by two parallel planes on either side of the equatorial plane and equidistant from it is worked out in detail. As a further extension elliptic coordinates are introduced and an analogous solution obtained, applicable throughout the volume of a hyperboloid of revolution of small transverse axis.
M. C. Gray (New York, N. Y.).

Keller, E. G. Beat theory of non-linear circuits. J. Franklin Inst. **228**, 319–337 (1939). [MF 274]

A circuit containing a capacity, a resistance, an inductance, and a non-linear inductance is treated. By representing the magnetizing current of the non-linear inductance as a fifth degree odd polynomial in its flux, the problem is reduced to the differential equation

$$\frac{d^2x}{d\theta^2}+P_1(x)\frac{dx}{d\theta}+P_2(x)=F(\theta),$$

where P_1, P_2 are proper fifth degree odd polynomials in x, and $F(\theta)$ is a known function of θ. This differential equation is solved by the familiar parameter method. The x^3 and x^5 terms in P_1 and P_2 are multiplied respectively by powers of a parameter m, namely by m^2 and m^4 and x is then expanded in powers of m^2. The results are compared with curves obtained by means of integration performed on the differential analyzer by J. W. Butler and C. Concordia and described in the A. I. E. E. Transactions, August, 1937.
H. Poritsky (Schenectady, N. Y.).

Pipes, Louis A. Matrix theory of oscillatory networks. J. Appl. Phys. **10**, 849–860 (1939). [MF 611]

The consequent application of the matrix calculus to the solution of passive linear dynamical systems as formerly introduced by Duncan and Collar [Philos. Mag. (7) **19**, 197 (1935)] is here illustrated by treating the problem of the dissipationless—and subsequently, as an approximation, of the moderately damped—electrical network of finite degree of freedom, with given initial conditions and impressed electromotive forces. While both the matrix solution of the steady state and the way of satisfying the initial conditions in terms of the steady state and transient solutions are conventional, the gist of the method is the production of the transient solution, that is, of the normal frequencies and modes of vibrations. These are obtained asymptotically by an iterative multiplication process which can be considered as a matrix equivalent of a Graeffe method for numerical equations. It separates first the largest characteristic root (smallest natural frequency). Then from the orthogonal properties of the eigensolutions, the "reduced" dynamical system is obtained which does not contain the first mode, and the repeated application of the iterative process leads consecutively to all the natural frequencies and to the total "modal matrix." The important point is that each step produces simultaneously the eigenvalue and the allied mode of vibrations ("modal column"), and it is mainly due to this fact that, for systems of many degrees of freedom, the matrix method is much superior to both the methods of classical dynamics and those based on and developed from Heaviside's calculus.

As an illustration, a numerical example is given which, however, in the referee's opinion, is too simple to be convincing, as it could be treated just as easily by conventional methods. More serious seemed the omission of "degenerate" systems which contain multiple eigenfrequencies. In practice, the method as developed does not apply to a proximity of natural frequencies where its convergence becomes very slow, and it must be generalized (in analogy to the corresponding adaptation of Graeffe's method) according to the treatment of Duncan and Collar [loc. cit.].
H. G. Baerwald (Cleveland, Ohio).

Schelkunoff, S. A. A general radiation formula. Proc. I. R. E. **27**, 660–666 (1939). [MF 741]

The classical (Hertzian) method of calculating the power radiated in nondissipative media by means of integration of the Poynting vector over distant surfaces ("radiation field") is condensed in a concise radiation formula which is dual with respect to electric and magnetic currents. This is advantageous in many applications where field distributions in certain characteristic parts of the space, equivalent to electric and magnetic current sheets, are known with a higher degree of approximation, from physical considerations, than the exciting current distributions. Some elementary examples are given. *H. G. Baerwald.*

Fues, E. Die Ausbreitungsfläche skalarer Wellen im gitterartigen Medium. Ann. Physik **36**, 209–226 (1939). [MF 401]

This is a continuation of earlier work of the author [Z. Phys. **109**, 14 and 236 (1938)]. A general treatment is given of wave propagation in lattice media, which do not much deviate from continuous media, in a non-analytical fashion by way of geometrical visualization based on the conception of the complex wave surface and the discussion of its topological and orientational properties. After treating the construction of the wave surface for different multiplicities of "strong" propagation inside the infinite lattice, the boundary problem of two media is taken up, first for the Laue case and then for the Bragg case of two rays. Finally the special conditions connected with the diffraction of fast electrons are discussed on the same basis, with emphasis on the interpretation of strong deformations and of the envelope phenomena of Kikuchi lines.
H. G. Baerwald (Cleveland, Ohio).

Hulme, H. R. Note on the integration of the equation of the formation of absorption lines. Monthly Not. Roy. Astr. Soc. **99**, 730–732 (1939). [MF 481]

The author presents two methods for the solution of

Eddington's equations

$$\frac{dH_r'}{d\tau}=(1+\epsilon\eta)(J_r'-J_r),\quad \frac{dJ_r'}{d\tau}=3(1+\eta)H_r',$$

for the formation of absorption lines in stellar spectra. (1) In the special case where

$$\epsilon=0,\quad J_r=H(2+3\tau),\quad 1/(1+\eta)=\alpha+\beta\tau+\gamma\tau^2,$$

a solution can be found in terms of hypergeometric functions. (2) In the general case where ϵ, J_r and η are given functions of τ, the equations can be solved by a differential analyser which has two input tables and three integrator units. *W. E. Milne* (Corvallis, Ore.).

Riabouchinsky, Dimitri. Quelques nouvelles remarques sur l'analogie supersonique du champ électromagnétique. C. R. Acad. Sci. Paris **209**, 587–589 (1939). [MF 517]
Concerning C. R. Acad. Sci. Paris **207**, 695–698 (1938).

Hoyle, F. Quantum electrodynamics. I and II. Proc. Cambridge Philos. Soc. **35**, 419–462 (1939). [MF 547, 548]

In the quantization of the usual radiation theory, square matrices $A_0, A_1, A_2, A_3, \psi_1, \psi_2, \psi_3, \psi_4$ are considered which all have the same number of elements depending upon the space-time coordinates x, y, z, t in such a way that, in a Lorentz transformation, (A_0, A_1, A_2, A_3) transform like the (t, x, y, z) components of a covariant vector and are real, while $(\psi^*\alpha_i\psi)$ $(i=0, 1, 2, 3)$ are the components of a contravariant vector, where $\psi^*=(\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$, ψ_s^* being the Hermitian conjugate of ψ_s, ψ is a column matrix with constituents $\psi_1, \psi_2, \psi_3, \psi_4$;

$$\alpha_0=1,\quad \alpha_i=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix},\qquad i=1, 2, 3,$$
$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The laws of variation for the matrices in space time are, if $D_t=\partial/\partial t$, etc.,

$$(1)\quad i\hbar D_t+eA_0(x, t)$$
$$+\sum_{r=1}^{3}(-i\hbar cD_{x_r}+eA_r(x, t))\alpha_r+\alpha_m mc^2]\psi(x, t)=0,$$

together with the conjugate equation, where α_m is a diagonal matrix with constituents $(1, 1, -1, -1)$, and the equations

$$(2)\quad [c^{-2}D_t^2-D_x^2-D_y^2-D_z^2]A_\mu(x,t)=\pm 4\pi e\psi^*(x,t)\alpha_\mu\psi(x,t),$$

with the plus sign for $\mu=1, 2, 3$ and the minus sign for $\mu=0$. With the same convention the commutation relations

$$\begin{aligned}(3)\qquad &[\psi_k(x, t)^*, \psi_{k'}(x', t)]_+=\delta_{kk'}\delta(x-x'),\\ &[A_\mu(x, t), D_tA_\nu(x', t)]_+=\pm 4\pi c^2 i\hbar\delta_{\mu\nu}\delta(x-x'),\\ &[D_tA_\mu(x, t), D_tA_\nu(x', t)]_-=[\psi_k(x, t), \psi_{k'}(x', t)]_+\\ &\quad=[\psi_k(x, t), A_\nu(x', t)]_-=[\psi_k(x, t), D_tA_\nu(x', t)]=0\end{aligned}$$

are shown to be consistent with (1) and (2). They are also shown to be invariant under a Lorentz transformation. To obtain physical results the matrices are expanded in Fourier series; the consistency of the commutation relations is then proved for the Fourier components. The restriction is next introduced that only those portions of the matrices A_0, A_1, A_2, A_3 that satisfy the relation $D_tA_0+c\operatorname{div} A=0$ shall enter into the physical interpretation of those matrices. An integral equation is then set up for $\psi(x, t)$ and attention is called to some mathematical difficulties which are associated to some extent with the supposition that the method of successive approximations can be applied to the integral equation. Two modifications of the Maxwell-Dirac field equations are then proposed. In the first the terms

$$\sum_{r=1}^{3}eA_r(x, t)\alpha_r\psi(x, t),\quad \pm 4\pi e\psi^*(x, t)\alpha_\mu\psi(x, t)$$

in (1) and (2) are replaced by

$$e\sum_{\mu=0}^{3}\bar{A}_\mu(x, t)\alpha_\mu\bar{\psi}(x, t),\quad \pm 4\pi e\bar{\psi}(x, t)^*\alpha_\mu\bar{\psi}(x, t),$$

respectively. In the second they are replaced by

$$e\sum_{\mu=0}^{3}\bar{A}_\mu(x, t)\alpha_\mu\psi(x, t),\quad \pm 4\pi e\psi(x, t)^*\alpha_\mu\psi(x, t),$$

respectively. A four-dimensional region of interaction between the radiation field and the material systems is thus postulated and this is shown to be in accordance with the principle of relativity.

In the second paper the quantization is considered for the new theory. Commutation relations are formulated for the Fourier coefficients and are shown to be consistent under certain conditions which seem to hold for atomic particles with velocities small compared with that of light but not for high energy particles in the form of wave-packets. The method of successive approximations is again considered, particular attention being paid to the first two approximations and the question of convergence. It is concluded that satisfactory results can be obtained when the space dimensions of the finite region of interaction are of the order of the classical radius of an electron at rest.
H. Bateman (Pasadena, Calif.).

Dirac, P. A. M. La théorie de l'électron et du champ électromagnétique. Ann. Inst. H. Poincaré **9**, 13–49 (1939). [MF 378]

Lorentz assumed the electron to be a small charged sphere whose mass and energy are due entirely to the electromagnetic field associated with it. A point model of the electron would be more convenient for quantum mechanics, which must start with a classical model. The Lorentz theory has not been able to account for the mass of the neutron or positron, nor to predict the behavior of rapidly accelerated electrons. This paper starts with an elaboration of an alternative classical (relativistic but not quantum theoretical) theory of the interaction between electrons and an electromagnetic field, recently proposed by the author [Proc. Roy. Soc. London, Ser. A. **167**, 148–169 (1938)]. In this new theory the electron, though merely a point, has associated with it a sphere of the same magnitude as the Lorentz sphere, through the interior of which a signal may be transmitted faster than light. The field due to the electron is not discontinuous at the surface of this sphere, but satisfies the field equations up to the center of the electron, where it has an infinite singularity. The equations of motion on this theory are shown to have the same form as those deduced from the theory of Lorentz although they are exact instead of approximate, and the physical interpretation is different.

The main purpose of the present paper is to develop the consequences for quantum mechanics of this new classical theory. It is first necessary to have equations of motion which are in the Hamiltonian form. To obtain these, the author changes slightly his previous theory, using instead of a Maxwellian field the potentials studied by Wentzel [Z. Phys. **86**, 479–494 (1933)], modified by the introduction of a small time-like vector λ, which is later to approach zero. Because of the presence of λ, the Hamiltonian equations are no longer invariant under the Lorentz group, but

are approximately so for small λ. Since the divergence of the potential of Wentzel does not vanish, the Schrödinger-Dirac wave equations derived from these Hamiltonian equations allow both longitudinal and transverse waves. When the longitudinal and transverse parts are separated, the wave equations take the form

$$\{[\mathbf{p}_n - e_n(\mathbf{B}(\mathbf{z}_n) + \mathbf{L}(\mathbf{z}_n))]^2 - m_n\}\chi = 0,$$

one equation for each electron. Here $\mathbf{p}_n$ is the usual operator $(h/2\pi i)(\partial/\partial \mathbf{z}_n)$ corresponding to the momentum vector of the nth electron, while $\mathbf{B}$ and $\mathbf{L}$ are Dirac operators corresponding to the transverse and longitudinal parts of the potential. In his calculations the author uses throughout the δ "functions" introduced by him, and the relativistic Δ function of Pauli and Jordan, as a convenient means of representing the singularities which the potentials have near the world lines of the electrons. The wave equations for spinning electrons are also derived. It is shown that in both cases the equations contain a term which becomes infinite as λ approaches zero. However, this infinity, which is present also in the theory of Heisenberg and Pauli, can be eliminated here by adding to the Hamiltonian functions a quantity which depends only on λ. *O. Frink.*

Taub, A. H. Tensor equations equivalent to the Dirac equations. Ann. of Math. **40**, 937–947 (1939). [MF 332]

The spinor formalism allowing one to rewrite tensor equations in spinor form and vice versa is recalled. The author expresses Dirac's equations for particles with spin one half and one in tensor form, and Maxwell's equations in spinor form. Dirac's equations for particles with spin one are compared with Maxwell's and Proca's equations.

L. Infeld (Toronto, Ont.).

Fokker, A. D. Hamilton's canonical equations for the motion of wave groups. Physica **6**, 785–790 (1939). [MF 254]

Reversing the steps of Schrödinger, the author starts with a differential wave equation of quite general form, (1) $F(D_k)\Psi = 0$, where $D_t = \partial/\partial t$, $D_x = \partial/\partial x$, etc., and seeks the equations of motion of a wave group in the canonical form. Assuming a solution of (1) of the form $\Psi = a \exp(i\chi)$, where both a and χ are functions of the coordinates and time, and assuming that the partial derivatives of a are small in comparison with the derivatives χ_k, the author shows that, to a first approximation, $F(i\chi_k) = 0$. Combining this result with some formulas of Lorentz for the path of a wave group gives the following set of canonical equations, again as a first approximation,

$$(2)\qquad \begin{aligned} &\frac{dt}{du} = \frac{\partial F}{\partial i\chi_t}, \qquad \frac{dx}{du} = \frac{\partial F}{\partial i\chi_x}, \qquad \text{etc.;} \\ &\frac{di\chi_t}{du} = -\frac{\partial F}{\partial t}, \quad \frac{di\chi_x}{du} = -\frac{\partial F}{\partial x}, \quad \text{etc.} \end{aligned}$$

Here u is a parameter, x, y and z are the coordinates of the wave group, while the quantities χ_k, which take the place of the momenta, determine the direction of the wave front. It is shown that the canonical equations (2) have the same relation to the wave equation (1) as in the Schrödinger theory. *O. Frink, Jr.* (State College, Pa.).

Badarau, Gabriel. Sur la propagation des groupes d'ondes et les relations entre la mécanique classique et la mécanique ondulatoire. C. R. Acad. Sci. Paris **209**, 551–554 (1939). [MF 522]

In the theory of material waves of L. de Broglie, the motion of a corpuscle is represented by the motion of a group of waves, the group velocity being that of the corpuscle. In the absence of a field, it is easy to show that the motion of the wave group corresponding to a corpuscle obeys the laws of classical mechanics. In the present paper the author, using perturbation methods, seeks to extend this result to the case of a corpuscle moving in a field. He uses the perturbation theory of Dirac, in which the wave function $\Psi(x, y, z, t)$ is expanded in a series of wave functions of the unperturbed problem, with coefficients which are functions of time. Since he takes the unperturbed state to be the motion of a free corpuscle in the absence of a field, which leads to a continuous spectrum, Ψ must here be represented, not by a series, but by an integral with kernel depending on the field which is introduced as a perturbation. From this integral for Ψ, the motion of the group of waves is determined by the condition that the amplitude be constant. Formulas for the velocity vector of the wave group are given, and it is shown how the curvature of the path depends on the field. *O. Frink* (State College, Pa.).

Taub, A. H. Spinor equations for the meson and their solution when no field is present. Phys. Rev. **56**, 799–810 (1939). [MF 309]

The equations for a particle with mass m, charge e and spin one were formulated by Proca [J. Phys. Rad. **7**, 347 (1936)] in tensor form. The author reformulates these equations in a spinor form and discusses the difference between them and the equations proposed by Dirac for particles with spin one. The solution of Proca's equations without external field for a particle with mass m is obtained through a solution of Dirac's equations for two particles with masses m_1 and m_2 and spin one-half, where $m = m_1 + m_2$.

L. Infeld (Toronto, Ont.).

Kemmer, N. The particle aspect of meson theory. Proc. Roy. Soc. London. Ser. A. **173**, 91–116 (1939). [MF 849]

The field equations of the meson theory can be written in a form closely analogous to the Dirac wave equation for electrons, only with a different set of commutation rules for the entering operators [Duffin, Phys. Rev. **54**, 1114 (1938)]. It is shown that all particle properties of the mesons, such as momenta, spin, etc., can be defined with the help of these operators in an abstract manner. The algebraic properties of the operators are investigated and it is shown that the only two non-trivial irreducible representations (with ten and five rows, respectively) correspond to the vectorial and scalar (Klein-Gordon) meson field theories which thus seem to be the only possible theories of particles with spin values 1 and 0. The connection with other forms of the meson theory is studied in detail and also the non-relativistic limit of the theory is deduced.

L. W. Nordheim (Durham, N. C.).

Haseltine, W. R. The mutual interaction of plasma electrons. J. Math. Phys. Mass. Inst. Tech. **18**, 174–201 (1939). [MF 191]

The paper contains a thorough mathematical analysis of the stationary distribution of electrons in an electric field under the influence of collisions with surrounding atoms and between the electrons mutually. The conservation of energy and momentum entails a nonlinear integro-differential equation, which is solved approximately, the interaction of the electrons being described by means of a "shielded" Coulomb potential. The result, a modified Maxwell distri-

bution, seems to be in general accord with experiments in discharge tubes. *O. Klein* (Stockholm).

Peierls, R. Critical conditions in neutron multiplication. Proc. Cambridge Philos. Soc. **35**, 610–651 (1939). [MF 838]

Let $1/\alpha$ be the mean free path of a neutron in a substance, β the number of secondary or scattered neutrons which emerge from unit length of the neutron's path in directions other than that in which this neutron is travelling. Then, if at time t there are $n(x, y, z, t)$ neutrons per unit volume of a sphere of radius a, the number of scattered and secondary neutrons emerging from a volume element $dxdydz$ in a time dt' is $\beta v n(x', y', z', t')dx'dy'dz'dt'$, where v is the neutron velocity. When these neutrons have travelled a distance R their number will be reduced by a factor $e^{-\alpha R}$ and they will now fill a spherical shell of radius R and thickness vdt' uniformly. The total number of neutrons at x, y, z is found by integration and this leads to an equation

$$4\pi n(x, y, z, t) = \beta \int n(x', y', z', t-R/c)e^{-\alpha R}R^{-2}dx'dy'dz'$$

which has solutions of the form $n(x, y, z, t) = n(x, y, z)e^{-\lambda t}$. The author is interested in the lowest possible value of the time constant λ, for when it is zero conditions just cease to be stable. The equation for n contains α and λ only in the combination $\alpha-\lambda/v$; consequently, if the integral equation obtained by putting $\lambda=0$ can be solved for all values of α the solution of the general equation can be derived. Perrin's solution of the integral equation by the methods of diffusion theory is extended to a higher order of approximation and a treatment is given of the case $\beta \gg \alpha$. The integral equation is then

$$f(X) = \beta a \int_0^1 Q_0(X'/X)f(X')dX', \qquad f(0)=0,$$

where $Q_0(z)$ is a Legendre function of the second kind. The kernel is symmetric and, though it is not bounded, its square is integrable and so a greatest eigenvalue exists. Limits for it are found by considering special functions f which are nonnegative. Some approximations and a method of interpolation are then given. *H. Bateman.*

van Dantzig, D. On the phenomenological thermodynamics of moving matter. Physica 6, 673–704 (1939). [MF 253]

The space-time of this paper has no metric, so that the quantities involved are tensor and scalar densities. All equations of the theory are invariant under a general point-transformation. The classical concept of a perfect fluid, as one in which the stress across every element is normal, has not the required invariance unless there is added the further condition that there is no transfer of heat by conduction. Then we have a fluid, which the author calls perfectly perfect, with the energy tensor

$$\mathfrak{P}_i^{\cdot h} = \pi_i \mathfrak{N}^h + \mathfrak{p}\delta_i^h,$$

where π_i is a vector of average momentum and enthalpy per particle, $\mathfrak{N}^h$ is a particle current vector (obtained by counting particles), and $\mathfrak{p}$ is a scalar pressure. On putting $\pi_i = \pi_0 i_i$, $\mathfrak{N}^h = \mathfrak{N}_0 i^h$, $-\pi_0\mathfrak{N}_0 = \rho + \mathfrak{p}$, this more general energy tensor reduces to the relativistic energy tensor, i^h being the velocity 4-vector. Entropy appears as the entropy current $\mathfrak{S}^h$ and temperature as the temperature vector θ^h. For a chemically homogeneous substance there is a scalar λ corresponding to the chemical potential. The entropy current is then $\mathfrak{S}^h = -(\lambda+\pi_i\theta^i)\mathfrak{N}^h$. The six quantities λ, $\mathfrak{p}$, θ^h characterize a state and an equation of state defines $\mathfrak{p}$ in terms of the five independent quantities λ, θ^h. In terms of them

$$\mathfrak{N}^h = \frac{\partial}{\partial\lambda}(\mathfrak{p}\theta^h), \quad \pi_i = \frac{\partial\mathfrak{p}}{\partial\theta^i}\Big/\frac{\partial\mathfrak{p}}{\partial\lambda}, \quad \mathfrak{P}_i^{\cdot h} = \frac{\partial}{\partial\theta^i}(\mathfrak{p}\theta^h).$$

The paper includes a discussion of specific heats.
J. L. Synge (Toronto, Ont.).

van Dantzig, D. On relativistic thermodynamics. Nederl. Akad. Wetensch., Proc. **42**, 601–607 (1939). [MF 315]

A brief account of the paper reviewed above.
J. L. Synge (Toronto, Ont.).

van Dantzig, D. On relativistic gas theory. Nederl. Akad. Wetensch., Proc. **42**, 608–625 (1939). [MF 316]

The method is very general, some of the results being independent of metrical geometry and valid in classical as well as relativistic theory. The main difference between the present treatment and those of Jüttner, Tolman and others lies in the symmetrical treatment of the time coordinate with the other coordinates. An original technique for the treatment of integrals in phase-space is introduced. The momentum-energy vector of a particle defines a point necessarily situated on a characteristic 3-space X ($H=0$) immersed in the momentum-energy 4-space K. To avoid integrals taken over X, the author extends functions defined over X throughout the portion of K lying on one side of X, the extensions vanishing exponentially at infinity, and so is able to convert integrals over X into integrals over part of K. The pressure in a gas is $\mathfrak{p} = h\int f d\mathfrak{k}$, where h is Planck's constant, f a distribution function and $d\mathfrak{k}$ an element of K. The energy tensor is

$$\mathfrak{P}_{\cdot\lambda}^{\ \kappa} = \mathfrak{p}\delta_\lambda^\kappa + h\int \kappa_\lambda \frac{\partial f}{\partial \kappa_\kappa} d\mathfrak{k}.$$

Here $\kappa_\lambda = p_\lambda/h$, where p_λ is the momentum-energy vector. When the Hamiltonian H (defining the dynamical properties of the molecules) and $f(\kappa_\lambda, q^\kappa)$ (defining the kind of statistics) are given, $\mathfrak{p}$ is at once obtained. All thermodynamic quantities can be found from $\mathfrak{p}$ by differentiation and algebraic processes. For a monatomic gas with Maxwell-Boltzmann statistics the author obtains the energy tensor

$$\mathfrak{P}_{\cdot i}^{\ h} = -\mathfrak{p}(\tau+5/2-\mathfrak{g}'\tau/\mathfrak{g})i_i i^h + \mathfrak{p}\delta_i^h,$$

where $\tau = mc^2/kT$ and $\mathfrak{g}(\tau) = 1+15/(8\tau)+\cdots$. Comparison with the usual relativistic energy tensor leads to the conclusion that density is given in terms of pressure and temperature by

$$\rho = \mathfrak{p}(\tau+3/2-\mathfrak{g}'\tau/\mathfrak{g}).$$

On the basis of this formula the author criticizes the definitions of density employed by Schwarzschild, Eddington and Synge. The paper ends with a discussion of black-body radiation. *J. L. Synge* (Toronto, Ont.).

AUTHOR INDEX

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